1 Administrative

- PS3 deadline is extended to March 24/25
- Project proposal is 1-4 pages

2 Smoothness ⇒ Bayesian Price of Anarchy

Auction game is \((\lambda, \mu)\) smooth if for fixed \(v\), \(\exists s^*(v)\), s.t \(\forall s\) (any),

\[
\sum_i u_i(s^*_i(v), s_{-i}) \geq \lambda \text{OPT}(v) - \mu \sum_i p_i(s)
\]

- Bayesian values \(\in\) distribution
- \(u^v_i(s)\) = utility of \(i\) when value is \(v_i\); \(v_i\) can be a vector
- \(\text{OPT}(v) = \max\ \text{SW}\) when values are \(v\)
- \(u^v_i(s^*_i, s_{-i})\) depends on \(v_i\)
- \(s^*_i\) depends on values \(v\): \(s^*(v)\)

Theorem 1. If \(\exists s^*(v)\), and auction is \((\lambda, \mu)\) smooth and \(s^*_i\) depends only on \(v_i\) (and not on \(v_{-i}\)), then

\[
\mathbb{E}(\text{SW(Nash)}) \geq \frac{\lambda}{\max \{1, \mu\}} \mathbb{E}(\text{OPT}(v))
\]

Example smooth games:

- \(s^*_i(v_i)\): first price single item
- \(s^*_i(v)\): \(\begin{cases} \text{all pay} & \text{price with multiple item and unit demand} \end{cases}\)

Today:
Theorem 2. If an auction is \((\lambda, \mu)\) smooth (even if \(s'_i\) depends on all coordinates of \(v\)), and the distribution of values for different players is independent, then:

\[
\mathbb{E}(SW(Bayesian\text{Nash})) \geq \frac{\lambda}{\max\{1, \mu\}} \mathbb{E}_v(\text{OPT}(v))
\]

- values to different items of a single bidder can be correlated
- values to items of different bidders cannot be correlated
- common knowledge: the distribution of values, as well as the strategies used at Bayesian Nash \(s_i(v_i)\), i.e., \(s_i\) as a function of \(v_i\), is common knowledge.
- if \(s\) is Bayesian Nash, then for all \(i\) and \(s'_i\) and all \(v_i\),

\[
\mathbb{E}_{v_{-i}}(u^v_i(s_i(v_i), s_{-i}(v_{-i}))|v_i) \geq \mathbb{E}_{v_{-i}}(u^v_i(s'_i, s_{-i}(v_{-i}))|v_i)
\]

An example of Bayesian Nash: 2 bidders, uniform \([0,1]\) distribution, and first price auction, \(b_i(v_i) = v_i/2\).

Proof. of the Theorem.

Take \(w_{-i}\) from value distribution of \(v_{-i}\); take \(s'_i(v_i, w_{-i})\), and use this as \(s'_i\). At a Bayesian Nash equilibrium

\[
\mathbb{E}_{v_{-i}}(u^v_i(s_i(v_i))|v_i) \geq \mathbb{E}_{v_{-i}, w_{-i}}(u^v_i(s'_i(v_i, w_{-i}), s_{-i}(v_{-i}))|v_i)
\]

Taking also expectation over \(v_i\) we get:

\[
\mathbb{E}_v(u^v_i(s(v))|v_i) \geq \mathbb{E}_{v, w_{-i}}(u^v_i(s'_i(v_i, w_{-i}), s_{-i}(v_{-i})))
\]

Sum up,

\[
\mathbb{E}_v(\sum_i u^v_i(s(v))) = \sum_i \mathbb{E}_v(u^v_i(s)) \geq \sum_i \mathbb{E}_{v, w_{-i}}(u^v_i(s'_i(v_i, w_{-i}), s_{-i}(v_i)))
\]

\((v_i, w_{-i})\) is of random draw of the type \(v\), because the different coordinates are independent. Define a new variable \(t = (v_i, w_{-i})\) as a phantom player, or simply as renaming of the variables \((v_i, w_{-i})\), and let \(z = (w_i, v_{-i})\) using a new random variable \(w_i\). Using the new variables \(t\) and \(z\) we can rewrite our sum as follows.

\[
\sum_i \mathbb{E}_{v, w_{-i}}(u^v_i(s'_i(v_i, w_{-i}), s_{-i}(v_i))) = \sum_i \mathbb{E}_{t, z}(u^v_i(s^*(t), s_{-i}(z))) \geq \mathbb{E}_{z, t}(\lambda \text{OPT}(t) - \mu \sum_i p_i(s(z)))
\]

\[
= \lambda \mathbb{E}_t(\text{OPT}(t)) - \mu \mathbb{E}_z(\sum_i p_i(s(z)))
\]
⇒ \( \mathbb{E}_v(\sum_i u_i^v(s(v))) \geq \lambda \mathbb{E}_t(\text{OPT}(t)) - \mu \mathbb{E}_z(\sum_i p_i(s(z))) \)

\[
\mathbb{E}_v(\text{SW}(\text{Nash})) = \mathbb{E}_v(\sum_i u_i^v(s(v))) + \mathbb{E}_v(\sum_i p_i(s(v))) \leq \frac{\lambda}{\max(1, \mu)} \mathbb{E}_z(\text{SW}(s(z))) \quad \square
\]