CS6840 - Algorithmic Game Theory (3 pages) Spring 2014

March 21 - Bayesian Price of Anarchy in Smooth Auction

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1 Administrative

- PS3 deadline is extended to March 24/25
- Project proposal is 1-4 pages

2 Smoothness ⇒ Bayesian Price of Anarchy

Auction game is (λ, μ) smooth if for fixed v, $\exists\ s^*(v)$, s.t $\forall s$ (any),

$$\sum_i u_i(s_i^*(v), s_{-i}) \geq \lambda \, \mathrm{OPT}(v) - \mu \sum_i p_i(s)$$

- Bayesian values ∈ distribution
- $u_i^{v_i}(s)$ = utility of i when value is v_i; v_i can be a vector
- OPT(v) = max SW when values are v
- $u_i^{v_i}(s_i^*, s_{-i})$ depends on v_i
- s^* depends on values v: $s^*(v)$

Theorem 1. If $\exists\ s^*(v)$, and auction is (λ, μ) smooth and s_i^* depends only on v_i (and not on v_{-i}, then

$$\mathbb{E}(\underbrace{SW(Nash)}_{a\ Bayesian\ Nash}) \geq \frac{\lambda}{\max\{1, \mu\}} \mathbb{E}_v(\mathrm{OPT}(v))$$

Example smooth games:

- $s_i^*(v_i)$: first price single item
- $s_i^*(v) : \begin{cases} all\ pay \\ price\ with\ multiple\ item\ and\ unit\ demand \end{cases}$

Today:

Theorem 2. if an auction is (λ, μ) smooth (even if s_i^* depends on all coordinates of v), and the distribution of values for different players is independent, then:

$$\mathbb{E}(SW(BayesianNash)) \geq \frac{\lambda}{\max\{1, \mu\}} \mathbb{E}_v(\text{OPT}(v))$$

- values to different items of a single bidder can be correlated
- values to items of different bidders cannot be correlated
- common knowledge: the distribution of values, as well as the strategies used at Bayesian Nash $s_i(v_i)$, i.e., s_i as a function of v_i, is common knowledge.
- if s is Bayesian Nash, then for all i and s_i' and all v_i,

$$\mathbb{E}_{v_{-i}}(u_i^{v_i}(s_i(v_i), s_{-i}(v_{-i}))|v_i) \geq \mathbb{E}_{v_{-i}}(u_i^{v_i}(s_i', s_{-i}(v_{-i}))|v_i)$$

  An example of Bayesian Nash: 2 bidders, uniform [0,1] distribution, and first price auction, $b_i(v_i) = v_i/2$.

Proof. of the Theorem.
take w_{-i} from value distribution of v_{-i}; take $s_i^*(v_i, w_{-i})$, and use this as s_i'. At a Bayesian Nash equilibrium

$$\mathbb{E}_{v_{-i}}(u_i^{v_i}(s)|v_i) \geq \mathbb{E}_{v_{-i},w_{-i}}(u_i^{v_i}(s_i^*(v_i, w_{-i}), s_{-i}(v_{-i}))|v_i)$$

Taking also expectation over v_i we get:

$$\mathbb{E}_v(u_i^{v_i}(s(v))) \geq \mathbb{E}_{v,w_{-i}}(u_i^{v_i}(s_i^*(v_i, w_{-i}), s_{-i}(v_{-i})))$$

sum up,

$$\mathbb{E}_v(\sum_i u_i^{v_i}(s(v))) = \sum_i \mathbb{E}_v(u_i^{v_i}(s)) \underbrace{\geq}_{Nash} \sum_i \mathbb{E}_{v,w_{-i}}(u_i^{v_i}(s_i^*(v_i, w_{-i}), s_{-i}(v_i)))$$

(v_i, w_{-i}) is of random draw of the type v, because the different coordinates are independent. Define a new variable $t = (v_i, w_{-i})$ as a phantom player, or simply as renaming of the variables (v_i, w_{-i}), and let $z = (w_i, v_{-i})$ using a new random variable w_i. Using the new variables t and z we can rewrite our sum as follows.

$$\sum_i \mathbb{E}_{v,w_{-i}}(u_i^{v_i}(s_i^*(v_i, w_{-i}), s_{-i}(v_i))) = \sum_i \mathbb{E}_{t,z}(u_i^{t_i}(s^*(t), s_{-i}(z))) \underbrace{\geq}_{smoothness} \mathbb{E}_{z,t}(\lambda\,\text{OPT}(t) - \mu \sum_i p_i(s(z)))$$
$$= \lambda\,\mathbb{E}_t(\text{OPT}(t)) - \mu\,\mathbb{E}_z(\sum_i p_i(s(z)))$$

$$\Rightarrow \; \mathbb{E}_v(\sum_i u_i^{v_i}(s(v))) \geq \lambda \, \mathbb{E}_t(\mathrm{OPT}(t)) - \mu \, \mathbb{E}_z(\sum_i p_i(s(z)))$$

$$\mathbb{E}_v(SW(Nash)) = \mathbb{E}_v(\sum_i u_i^{v_i}(s(v))) + \mathbb{E}_v(\sum_i p_i(s(v))) \leq \frac{\lambda}{\max(1,\mu)} \, \mathbb{E}_z(SW(s(z))) \quad \square$$