1 Review:

Definition. An auction is \((\lambda, \mu)\)-smooth if there exists \(s^*\), s.t. for all \(s\):
\[
\sum_i u_i(s_i^*, s_{-i}) \geq \lambda \text{OPT} - \mu \sum_i p_i(s).
\]

Smooth auctions: Set up:
- \(o(s)\): outcome
- \(v_i(o)\): value of player \(i\). \(\text{OPT} = \max_o \sum v_i(o)\)
- \(u_i(s) = v_i(o(s)) - p_i(s)\)
- \(p_i(s) = \text{ith payment}\)

Last Time: Smoothness for single item 1st price auction.

Theorem 1. All pay single item auction is \((\frac{1}{2}, 1)\)-smooth for any distribution of values.

Proof. Let \(i^* = \arg \max_i V_i\). Let \(s_j^* = 0\) for \(j \neq i^*\) and \(s_i^*\): randomly chosen according to uniform distribution in \([0, v_i]\). For \(j \neq i^*\):
\[
u_j(s_j^*, s_{-j}) \geq 0;
\]
for \(j = i^*\), let \(p = \max_{j \neq i^*} s_j\), then:
\[
u_{i^*}(s_i^*, s_{-i^*}) \geq -E(s_i^*) + v_{i^*}Pr(i^* \text{ wins})
= -\frac{v_{i^*}}{2} + v_{i^*} \left( \frac{v_{i^*} - p}{v_{i^*}} \right)
= 0.5v_{i^*} - p
\geq 0.5v_{i^*} - \sum_j p_j(s)
\]
Sum up over all \(i\), we get:
\[
\sum_i u_i(s_i^*, s_{-i}) \geq \frac{1}{2} \text{OPT} - \sum_i p_i(s)
\]

2 Multiple Items:

2.1 Set up for today:
- Unit demand bidders
• Items on sale: $\Omega$
• Players: $1, \ldots, n$
• Player $i$ has value $v_{ij} \geq 0$ for item $j$
• $A \subset \Omega$, player $i$'s value for set $A \neq \emptyset$ is $\max_{j \in A} v_{ij}$ (there is free disposal).

2.2 Smoothness

Today: each item is sold on first price.

VCG Mechanism: uses OPT assignment. First price auction uses opt assignment in analysis, but not on mechanism.

Max value matching (optimal matching): $\max_M \sum_{(i,j) \in M} v_{ij}$, $M$ represents a Matching.

**Theorem 2.** 1st price multiple items auction is $(\frac{1}{2}, 1)$-smooth (also $(1 - \frac{1}{e}, 1)$-smooth).

**Proof.** Take optimal matching $M^*$. If $(i, j) \in M$ (player $i$, item $j$), then bid $s^{*i}_{j} = \frac{v_{ij}}{2}$ for item $j$ and bid 0 for all other items. If $i$ is unmatched in $M$, bid 0 on all items.

If $i$ unmatched,  
$$u_i(s^*_i, s_{-i}) \geq 0;$$

Else, $(i, j) \in M$,  
$$u_i(s^*_i, s_{-i}) \geq \frac{v_{ij}}{2} - p_j(s).$$

$p_j(s)$ is price for item $j$ on bids $s$. (This is because if player $i$ wins item $j$, $u_i(s^*_i, s_{-i}) = \frac{v_{ij}}{2}$; if player $i$ loses item $j$, item $j$'s price $p_j(s)$ is $\geq \frac{v_{ij}}{2}$.) Sum over $i$:

$$\sum_i u_i(s^*_i, s_{-i}) \geq \frac{1}{2} \sum_{(i,j) \in M} v_{ij} - \sum_{j \in A} p_j(s) = \frac{1}{2} OPT - \sum_j p_j(s)$$

($p_j = 0$ if item $j$ not in assigned).

**Corollary 3.** Nash equilibrium $s$ for full information game satisfies:

$$SW(s) \geq \frac{\lambda}{\max\{1, \mu\}} OPT.$$ 

Want Bayesian version:

Option 1: $s^*_i$ depends only on $v_i$ (ith valuation). We used it in single item 1st price auction. Doesn’t apply to either "all-pay" of auctions with multiple items.
3 Next Time:

Theorem: smooth game $\rightarrow$ Bayesian PoA small

2nd price auction