CS6840 - Algorithmic Game Theory (2 pages) Spring 2014

April 11 - Complement Free Valuations

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Classes of valuations

We started to consider three classes of valuations last time. For a set A, we will use $v(A)$ to be the value of set A to a user. We will not index valuations with users this class, as we will only consider one user. For all classes we consider today, we will assume that $v(\emptyset) = 0$, value is monotone, that is $A \subset B$ implies that $v(A) \leq v(B)$ (there is free disposal). Note that this also implies that $v(A) \geq 0$ for all A.

1. subadditive valuations, requiring that for any pair of disjoint sets X and Y we have $v(X) + v(Y) \geq v(X \cup Y)$.
2. diminishing marginal value, requiring that for any element j and any pair of sets $S \subset S'$ we have $v(S + j) - v(S) \geq v(S' + j) - v(S')$
3. fractionally subadditive: defined as a function v obtained from a set of vectors v^k with coordinates v^k_j for some $k = 1, ..$ with $v(A) = \max_k \sum_{j \in A} v^k_j$.

First we want to show that diminishing marginal value has the following alternate definition called submodular. A function is submodular, if for any two sets A and B the following holds.

$$v(A) + v(B) \geq v(A \cap B) + v(A \cup B).$$

Claim 1. A function v that is nonnegative, monotone, and $v(\emptyset) = 0$, it is submodular if and only if it satisfies the diminishing marginal value property.

Proof. First, we show by induction that for a pair of sets $S \subset S'$, and a any set A the following diminishing marginal value property holds $v(S \cup A) - v(S) \geq v(S' \cup A) - v(S')$. We show this by induction on $|A|$. When $|A| = 1$ this is the diminishing marginal value property. When $A = A' + j$, by the induction hypothesis $v(S \cup A') - v(S) \geq v(S' \cup A') - v(S')$, by the diminishing marginal value property applied to $S \cup A' \subset S' \cup A'$, we get $v(S \cup A' + j) - v(S \cup A') \geq v(S' \cup A' + j) - v(S' \cup A')$. Adding the two we get $v(S \cup A) - v(S) \geq v(S' \cup A) - v(S')$ as claimed.

For sets $S \subset S'$ a set A disjoint from S', let $X = S \cup A$, and $Y = S'$ then the diminishing marginal value property is exactly the submodular property with X and Y, and vice versa, the submodular property for sets X and Y is this diminishing marginal value property with $S' = Y$, $S = X \cap Y$ and $A = X \setminus Y$. □

Next we show that all fractionally subadditive functions are subadditive.

Claim 2. A fractionally subadditive function is subadditive.

Proof. Let A and B two disjoint sets. The value $v(A \cap B) = \max_k \sum_{j \in A \cup B} v_j^k$. Let k^* be the value that takes the maximum. Now we have

$$v(A \cup B) = \sum_{j \in A \cup B} v_j^{k^*} = \max_k \sum_{j \in A} v_j^{k^*} + \max_{k^*} \sum_{j \in B} v_j^k \leq v(A) + v(B).$$

□

Claim 3. Any submodular function is fractionally subadditive.

Proof. For a submodular function v, we define vectors v_j^k that define v as a required for a fractionally subadditive function. For any order k of the elements, let B_j^k denote the set of first j elements of the order k. For ℓ's element in this order, $\{æ\} = B_\ell^k - B_{\ell-1}^k$, we define $v_j^k = v(B_\ell^k) - v(B_{\ell-1}^k)$. We claim that this defines v.

For a set A, and any order k that starts with A, clearly $v(A) = sum_{j \in A} v_j^k$.

We need to show that for all orders k we have $v(A) \leq \sum_{j \in A} v_j^k$. For this order k define the related order k' that is the same as k in ordering A, but has elements not in A after all elements of A. By the above $v(A) = \sum_{j \in A} v_j^{k'}$, and by the diminishing marginal value property $v_j^{k'} \leq v_j^k$ for all $j \in A$. □

Finally, we wonder about how many functions needed in defining a fractionally subadditive function, and which functions can be defined this way. For a vector v^k to be useable in the definition, it must satisfy $v_j^k \geq 0$ and $\sum_{j \in A} v_j^k \leq v(A)$ for all sets A. To be able to define a function v as fractionally subadditive, for all sets X we need such a vector v^k that also has $\sum_{j \in X} v_j^k = v(X)$. Looking for such a v^k can be written this as a linear program as follows:

$$x_j \geq 0 \text{for all } j \tag{1}$$

$$\sum_{j \in A} x_j \leq v(A) \text{for all sets } A \tag{2}$$

$$\sum_{j \in X} x_j = v(X) \tag{3}$$

A valuation v is fractionally subadditive, if and only if this linear program has a solution for all sets X. Note that this also shows that it suffices to have 2^n vectors v^k in the definition. To see the condition required for a function to be fractionally subadditive, one takes linear programming dual (or Farkas lemma) to get the condition needed to make the above linear program solvable.