CS 6840 – Algorithmic Game Theory — Friday, April 13, 2012

Lecture 35 Scribe Notes

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1 Overview

1.1 Summary

In this lecture, we:

- Analyze the Price of Anarchy of the fair-sharing model for bandwidth sharing along a single network edge (as introduced in the previous lecture). It turns out to be at most $\frac{4}{3}$.
- Introduce a new approach to bounding the PoA of a set of problems – we create a many-to-one mapping f from our set of problems into a more restricted subset of problems, such that f can only increase the PoA. We do this strategically so that it is easier to calculate the PoA of the subset. This is different from approaches focused on agent behavior.

2 Context

2.1 Recap of Bandwidth Fair-Sharing

Last lecture, we introduced a bandwidth sharing problem:

- n users want to share a *single edge* of a network.
- Users have utility $U_i(x)$ for bandwidth x, which is non-negative, monotone nondecreasing, concave, and differentiable.
- Users receive allocations x_i.
- The edge has total capacity $B = \sum_i x_i$.

We came up with the following fair-sharing allocation scheme:

- Each user comes up with a bid w_i representing his/her willingness to pay.
- Users pay their w_i.
- Users receive a fraction of the bandwidth proportional to their bids: $x_i = \left(\frac{w_i}{\sum_j w_j}\right) B$.
- If a user increases his bid, he will get more bandwidth, but also will increase $p_{eff} = \frac{\sum_j w_j}{B}$.

2.2 Key Results from Last Lecture

Last lecture, we found that:

- Price equilibrium: $U_i'(x_i) = p$, e.g. when i is allocated bandwidth until marginal payoff becomes zero.
- Proof that price equilibrium exists in the first place.
- Fair-sharing Nash Equilibrium: $U_i'(x_i)\left(1 - \frac{x_i}{B}\right) = p_{eff}$.
- In a case with many users, $\frac{x_i}{B} \approx 0$, so this mechanism is approximately optimal.

3 Price of Anarchy Analysis

3.1 Overview

As mentioned in the Summary, we will perform two steps which will map a given problem (specified by a set of U_i functions) to one whose PoA is strictly not worse. We will then be able to reason algebraically about the PoA of a simpler, restricted set of problems, and upper-bound the PoA in the general case.

The three steps are detailed in the following subsections. We'll use the notation: x_i is the allocation to i at Nash Equilibrium, x_i^* is the optimal (in the maximum sum-of-utilities sense) allocation, and use p to refer to p_{eff}.

3.2 Step 1: Map into the set of linear functions, $U_i(x) = a_i x_i + b_i$

Consider a corresponding problem in which each U_i is mapped to a new utility function V_i, which is the **tangent** to $U_i(x)$ at $x = x_i$, the Nash allocation. Explicitly, $V_i(x) = U_i'(x_i)(x - x_i) + U(x_i)$.

We see that the allocation $\vec{x}$ is still at Nash Equilibrium:

$$V'(x_i) = U'(x_i) \implies V_i'(x_i)\left(1 - \frac{x_i}{B}\right) = p_{eff}$$

Note that the optimal social value didn't get worse: Because U_i are concave, $V_i(x_i^*) \geq U_i(x_i^*)\ \forall i$, and thus there exists an allocation at least as socially optimal as the optimal allocation in the U problem.

Because the Nash value didn't change and the optimum didn't decrease, the PoA did not decrease.

3.3 Step 2: Map into the space of linear functions through $(0, 0)$, $Y_i(x) = a_i x_i$

Consider a correspondign problem in which each $V_i(x) = a_i x + b_i$ is mapped to a new utility function $Y_i(x) = a_i x$, which is $V_i(x)$ shifted to cross the origin. We'll show that the PoA in this restricted subset of problems is not improved.

First, observe that $b_i \geq 0$ must have been true, since $U_i(0) \geq 0$ by stipulation, and $b = V_i(0) \geq U_i(0)$ due to V_i never being less than U_i (a consequence of concavity). It follows that each of the V_i was shifted **down** to get W_i.

Now note that $Y'(x_i) = V'(x_i) = U'(x_i)$, so as before, the Nash allocation doesn't change.

From these, we see that the Nash social value decreases by $b_\Sigma = \sum_i b_i \geq 0$, and the optimal allocation must have decreased by the same amount (a vertical shift does not introduce a chance to improve the allocation).

Letting O and N be the respective total social values from utility functions V_i, we have:

$$\begin{aligned} O &\geq N \\ O \times N - N \times b_\Sigma &\geq O \times N - O \times b_\Sigma \\ N(O - b_\Sigma) &\geq O(N - b_\Sigma) \\ \frac{O - b_\Sigma}{N - b_\Sigma} &\geq \frac{O}{N} \end{aligned}$$

Thus, we see that the PoA has not decreased under this mapping.

3.4 Step 3: Bound the worst PoA in the restricted problem space

At this point, we note that the socially optimal allocation awards the entire bandwith to the user with the highest a_i. For convenience, we'll sort all users by a_i, so that the optimal allocation gives B to a_1, for a total optimal utility of $O' = Ba_1$.

The sum of utilities at Nash, on the other hand, is $N' = \sum_i Y_i(x_i) = \sum_i a_i x_i = a_1 x_1 + \sum_{i>1} a_i x_i$.

Note that if $a_i \leq p$, then in the Nash allocation, $x_i = 0$, so only people with $a_i > 0$ contribute to decreased social welfare. We'll use this fact to, holding the optimal value constant now (instead of the Nash), make the Nash value as poor as possible.

Recall that, at equilibrium, $Y'(x_i) = a_i\left(1 - \frac{x_i}{B}\right) = p$. Rearranging this, we see that i's utility is $Y(x_i) = a_i x_i = B\,(a_i - p)$. To conceive a worst-case bound, we want to make this value as low as possible *while still allocating to* i, i.e. be as wasteful as possible of this capacity x_i, which was allocated to i rather than 1. So, the worst case bound comes from choosing a_i very close to p, that is, $a_i = p + \varepsilon$ for very small ε. It follows that px_i is a lower bound on $Y_i(x_i) = a_i x_i$, the Nash utility.

$$
\begin{aligned}
PoA &\le O'/N' \\
&= \frac{Ba_1}{a_1x_1 + \sum_{i>1} a_ix_i} \\
&\le \frac{Ba_1}{a_1x_1 + \sum_{i>1} px_i} \\
&= \frac{Ba_1}{a_1x_1 + p\left(\sum_{i>1} x_i\right)} \\
&= \frac{Ba_1}{a_1x_1 + p(B - x_1)} \\
&= \frac{Ba_1}{a_1x_1 + a_1\left(1 - \frac{x_1}{B}\right)(B - x_1)} \\
&= \frac{Ba_1}{a_1x_1 + a_1\left(1 - \frac{x_1}{B}\right)(B - x_1)} \\
&= \frac{B}{x_1 + (1 - \frac{x_1}{B})(B - x_1)} \\
&= \frac{1}{\frac{x_1}{B} + \left(1 - \frac{x_1}{B}\right)^2}
\end{aligned}
$$

Differentiating with respect to the ratio $\frac{x_1}{B}$, we find that our PoA upper bound occurs at $\frac{x_1}{B} = \frac{1}{2}$ via calculus, so that worst case PoA is:

$$
\boxed{PoA \le \frac{1}{\frac{1}{2} + \left(1 - \frac{1}{2}\right)^2} = \frac{4}{3}}
$$

Which is our final result.

4 Existence of Nash Equilibrium

Last lecture, we saw that there was necessarily a price equilibrium. As it turns out, an almost identical proof works to show that there exists a Nash Equilibrium. We can even reduce the proof of existence of a Nash Equilibrium to the same proof used for a price equilibrium:

- We seek to establish the existence of an allocation such that $U_i'(x_i)\left(1 - \frac{x_i}{B}\right) = p_{eff}$.
- Define 'effective' utility function whose derivative is $U_{i,eff}'(x_i) = U_i'(x_i)\left(1 - \frac{x_i}{B}\right)$. This can be found by integrating by parts.
- Note that $U_{i,eff}'$ is decreasing if $U_i'(x)$ were decreasing, since the multiplicative factor is decreasing in x_i, so our property of concavity is maintained.
- Since the multiplicative factor is > 0 for all $x_i < B$, the multiplicative factor is positive, and thus $U_{i,eff}'$ is positive.

- Thus, this 'effective' utility function has the properties we required of the actual utility function in our proof of the existence of a price equilibrium.

5 Overview of Next Lecture

- We'll introduce a network version of the problem, in which each user has a desired path through the network, bids for each edge $e \in$ his path, and receives bandwidth equal to the minimum of his bandwidth along any edge in the path.
- We'll analyze a mechanism in which we run fair-sharing on each edge individually.
- We'll show that the PoA of fair-sharing in the network game is also $\frac{4}{3}$