CS 6840 – Algorithmic Game Theory (3 pages) Spring 2012

Lecture 39: The Existence of Nash Equilibrium in Finite Games

Instructor: Eva Tardos *Wenlei Xie(wx49)*

Today we will use the same methodology that we used in the last time to prove Nash Equilibrium exists in finite games.

Theorem 1 (Existence of NE)**.** Game with finite set of players and finite strategy sets has at least one (mixed) Nash Equilibrium.

Remark (Finite Game and Mixed NE)**.** This only applies to what usually called *Finite Games*. Here "finite" means two things: finite set of players and each of them has finite set of strategies. So this theorem doesn't apply to a bunch of games we studied, e.g. when the strategy is the price, the strategy set is not finite since it can be real numbers. Some other games could have infinite players. In most of these games, we actually have other arguments to prove that NE exists, usually even a better argument because we used to prove a pure strategy NE exists. And this theorem only states a mixed strategy NE exists, which is not surprising because some small 2 by 2 games, e.g. Pennies Matching game doesn't have a pure NE.

To prove this theorem, the main tool we will use is the Brouwer fixpoint theorem.

Theorem 2 (Brouwer Fixpoint Theorem)**.** If C is bounded, convex and closed, and $f : C \to C$ is continuous, there exists x s.t. $f(x) = x$.

Remark. Last time we only did it for the simplex. Generally we certainly need it to be bounded and closed. Topologically, we can make stronger statements than convex – but convex is certainly enough for today.

We will start with a natural but problematic proof. What we want to do is the same story as last time. Starting from one possible game state, which is a set of mixed strategies of all the players, we would like to know if it's an equilibrium or not. And if it's not we want a function that moves it more "closer" to the equilibrium.

Let n be the number of players and S_i to be the strategy set of player i, and Δ_i be the probability distribution space of strategies for player i, i.e.

$$\Delta_i = \{(p_s : s \in S_i) \,|\, p_s \geq 0 \text{ and } \sum_{s \in S_i} p_s = 1\} \tag{1}$$

We use C to denote the set of the mixed strategies of all the players, i.e. $C = \Delta_1 \times \Delta_2 \times \cdots \times \Delta_n$. It can be proved that C is convex, bounded and closed. Next we need a function $f : C \to C$ that the NE is a fixpoint. A natural answer is to use the best response. That is to say, given $p = (p_1, p_2, \cdots, p_n) \in C$, where $p_i \in \Delta_i$. Let q_i be the best response of player i, we could define the function as $f(p) = (q_1, q_2, \cdots, q_n)$. This can be viewed as all the players are moving to the best response state simultaneously as if others don't move.

The fundamental issue in this "proof" is that f might not be a function since the best response for the player might not be unique. A natural way to address this issue is to use lexicographic tie-breaking rule. Unfortunately the function constructed in this way might not be continuous. Let's consider the following example.

	Heads	Tails
Heads	$(+1, -1)$	$(-1, +1)$
Tails	$(-1, +1)$	$(+1, -1)$

Table 1: The payoff matrix for the matching pennies game

Example (Matching Pennies)**.** Recall the payoff matrix in the matching pennies game shown in Table 1. Suppose the mixed strategy for the first player is $(p_1, 1-p_1)$, i.e. he will turn the penny into head with probability p_1 and turn it into tail with probability $1-p_1$. Then the best response $(q_2, 1-q_2)$ for the second player is

$$\text{Best Response} = \begin{cases} q_2 = 0 & \text{if } p_1 > 1/2, \\ q_2 = 1 & \text{if } p_1 < 1/2, \\ 0 \le q_2 \le 1 & \text{if } p_1 = 1/2. \end{cases}$$

And clearly it is not continuous at $p_1 = 1/2$.

Thus, we need some better methods to fix this issue. We will discuss two options in this lecture.

Option 1 (Set Function)**.** Define $f : C \to 2^C$ as $f(p) = \{q \,|\, q_i \text{ is best response for } p_{-i}\}$

In this option, we hope to find the p, s.t. $p \in f(p)$. To this end we need to use a stronger fixpoint theorem by Kakutani and formally define what does "continuity" means for such set functions. We are not going to discuss the details in today's lecture.

Option 2 (More Sophisticated Objective Function)**.** Let $u_i(q, p_{-i})$ be the utility of player i playing q in response to p_{-i}. Here comes the natural best response function

$$\max_q u_i(q, p_{-i}) \to \text{ Original best response } q$$

As we have shown before, this doesn't define a function, and the natural way to make it a function breaks the continuity. Alternatively, consider

$$\max_q u_i(q, p_{-i}) - ||p_i - q||^2$$

So for player i, instead of maximizing the utility $u_i(q, p_{-i})$, it maximizing the utility minus a penalty from going away from the original p_i, i.e. $||p_i - q||^2$. Notice any positive scale for $||p_i - q||^2$ works. Suppose the maximizer for player i is q_i, we define $f(p) = (q_1, q_2, \cdots, q_n)$.

To finish the proof, we first claim that it indeed defines a function, which means the maximizer is unique.

Lemma 3. $\max_q u_i(q, p_{-i}) - ||q - p_i||^2$ is unique.

To prove this, we will use the fact that strictly concave function has unique maximization. Notice there are many definitions for strictly concave for vector functions. And we will use the following definition in our proof.

Definition (Strictly Concave)**.** $g(x)$ is strictly concave of x if

$$\forall x, x', \ \frac{1}{2}(g(x) + g(x')) > g(\frac{x + x'}{2}) \tag{2}$$

Proof. Notice that
$$u_i(q, p_{-i}) = \sum_{s \in S_i} q_s v_s(p_{-i}) \tag{3}$$
where $v_s(p_{-i})$ is the value of pure strategy s. Thus $u_i(q, p_{-i})$ is a linear function of q. And $-||q-p_i||^2$ is a strictly concave function of q, which makes $u_i(q, p_{-i}) - ||q - p_i||^2$ strictly concave, and it has unique maximization. □

To show f is continuous, we will use the following fact from convex optimization without proof.

Claim 1. If a class of optimization problems has unique optima, then the optimum is a continuous function of the coefficients in the objective function

An important part that is missing is that we want to show the fixpoint of function f is the Nash. When f is defined by the maximizer of $u_i(q, p_{-i})$, it is obvious. Now with the penalty term it is less obvious, but we can nonetheless prove it.

Lemma 4. If $f(p) = p$, then p is Nash.

If p is not Nash, there is some other best response $q = (q_1, q_2, \cdots, q_n)$. For the player i that doesn't perform best response, consider move from p_i to q_i. It will certainly increase the first part of the objective function $u_i(q_i, p_{-i})$. However the whole objective function might not be increased since the second part $||q_i - p_i||^2$ is also increased. And we will show if we just move on that direction small enough, it will be OK.

Proof. Suppose p is not Nash. Suppose one best response is $q = (q_1, q_2, \cdots, q_n)$. For player i that p_i is not best response, we have
$$u_i(q_i, p_{-i}) > u_i(p_i, p_{-i}) \tag{4}$$

Let $r_i(\epsilon) = (1-\epsilon)p_i + \epsilon q_i$. And if player i move from p_i to $r_i(\epsilon)$, consider the change in his objective function $\delta_i(\epsilon)$

$$\begin{aligned}\delta_i(\epsilon) &= \Big(u_i(r_i(\epsilon), p_{-i}) - ||r_i(\epsilon) - p_i||^2\Big) - u_i(p_i, p_{-i}) \\ &= \epsilon\Big(u_i(q_i, p_{-i}) - u_i(p_i, p_{-i})\Big) - \epsilon^2 ||q_i - p_i||^2\end{aligned}$$

For small enough ϵ, we have the change $\delta_i(\epsilon) > 0$. Hence p_i does not maximize $u_i(q, p_{-i}) - ||q - p_i||^2$, which means p couldn't be a fixpoint. □

Remark (Function f)**.** The function f assumes everyone simultaneously best response, but how would people's utility change as everyone best response? We don't know. This is a proof and it wants to do this artificial yet somewhat meaningless activity, of considering a function where everyone best response as if the other people don't move. If they don't move, we reach the Nash. But if they move, it is meaningless. Notice it is not a game dynamic, in fact it is nothing but a mathematic gadget of proof.

In the following lectures, we will show if you can find the NE in a game, you can find the fixpoint of the corresponding function.