Review Last time, we discussed greedy algorithm as mechanism and generalized single item auction to matroid case when greedy is still optimal.

Today We'll look at combinatorial auctions, in which there are a set of items $S$ on sale and player $i$ has value $v_i(A) \geq 0$ for all subset $A \subseteq S$.

We can do VCG on this setting, but there are some troubles (within brackets) related to the procedure of VCG

1) Ask players to report fns $v_i(A)$ [If $S$ big, then too many values to report, e.g. $2^{\mid S\mid}$]

2) Find allocation $A_i$ s.t. $A_i \cap A_j = \emptyset$ which,

$$\max \sum v_i(A_i)$$

[NP-hard to compute the "set-packing" allocation problem]

3) Compute payment

Today we are going to focus on single minded bidders where players $i$ has value $v_i$ and set $A_i$, and player $i$ gets value $v_i$ if he receives any set containing $A_i$, and gets value 0 otherwise.

![Diagram](image)

Figure 1: The routing example

A similar example is the routing case, where $S =$ edges in graph $G$, player $i$ has value $v_i$ and some source-sink pair $s_i - t_i$, player $i$ gets value $v_i$ if $A_i$ contains $s_i - t_i$ path, and 0 otherwise.
Algorithm 1: The greedy algorithm framework

Start with $I = \emptyset$

while not all players have been processed do
    Select $v_i$ with max $v_i/\sqrt{|A_i|}$ or $v_i/\sqrt{d(s_i, t_i)}$ ($d(s_i, t_i)$ is the length of shortest path from $s_i$ to $t_i$ using only edges unassigned)
    Add to $I$ if possible ($A_i$ disjoint from sets in $I$)
end while

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(a) Bad example if we choose the max $v_i$, here $v_0 = 1+\epsilon$ and $v_i = 1$ for other $i$

(b) Bad example if we choose the max $v_i/|A_i|$

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Greedy algorithm: Is there a payment scheme making greedy truthful? When reporting $v_i$ is equilibrium? We need

- Results for players monotone in value
- Reporting higher $v_i$ cannot cause player to lose
- Reporting true $A_i$ is also best for winning.

payment that makes this truthful: critical value.

The algorithm assume sorted order and price determined by first set after $i$ in sorted order that is not allocated due to $A_i$.

**Theorem 1.** Greedy with $v_i/\sqrt{|A_i|}$ and critical value payment is truthful and $\sqrt{n}$-approximation for $|S| = n$.

**Proof.** To show this is truthful, first, the players have better report a superset of $A_i$. If reporting a set not containing $A_i$, then even if they win, they will get no value. If they report any set larger than $A_i$, this will only decrease their likelihood of winning. Thus, it’s truthful to report the true set $A_i$. The truthfulness of reporting value $v_i$ follows the same argument of second price’s truthfulness.

Now let’s show the mechanism is $\sqrt{n}$-approximation. Suppose the algorithm took $A_i, i \in I$ and the Opt took $A_j, j \in O$. Let $C_i$ be the set in Opt not taken due to $A_i$. Then we have $v_i/\sqrt{|A_i|} \geq v_j/\sqrt{|A_j|}$ for any $j \in C_i$,

$$\sum_{j \in C_i} v_j \leq \sum_{j \in C_i} \sqrt{|A_j|} \frac{v_i}{\sqrt{|A_i|}}$$

Since $|C_i| \leq |A_i|$ and $\sum_{j \in C_i} |A_j| \leq n$,

$$\frac{\sum_{j \in C_i} \sqrt{|A_j|}}{\sqrt{|A_i|}} v_i \leq \frac{v_i}{\sqrt{|A_i|}} (|A_i| \sqrt{\frac{n}{|A_i|}}) = \sqrt{n} v_i$$
Sum over all $i \in I$, we get

$$\sum_{i \in I} \sum_{j \in C_i} v_j \leq \sqrt{n} \sum_{i \in I} v_i$$

which concludes our proof. \qed