Lecture 18 Scribe Notes

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Lecture 18 – Monday 23 January 2012 - The VCG Mechanism

Overview/Review

The purpose of this lecture is to show the **VCG mechanism**, which generalizes the Vickery Auction to much more general settings. It does this by giving a pricing mechanism so that to make the auction **truthful** (i.e., each player's best strategy is to bid his true value).

Remember our definition of a single item auction:

- n players
- a value v_i for each player i
- goal: pick a winner i^* maximizing the social welfare $SW(i^*) = v_{i^*}$
- each player maximizes each own utility: $u_i = v_i - p$ if $i = i^*$ and $u_i = 0$ otherwise.

In this setting, we have a truthful auction: the Second Price Auction (or Vickery Auction):

- each player i submits a bid b_i to the auctioneer
- the player with the highest bid wins the auction ($i^* = \operatorname{Arg\,max} b_i$)
- player i^* pays the second highest bid

This is interesting because we had an optimization problem (find the maximum v_i) over unknown input. So we defined a game to solve it.

Vickery-Clarke Groves mechanism

We are interested in solving the following optimization problem:

- n players
- a set of *alternatives* A that we can perform
- player i has a value $v_i(a)$ for each $a \in A$
- If alternative a^* is selected, player i's utility is $u_i = v_i(a^*) - p_i$
- goal is to select the alternative A^* that maximizes the social welfare: $\sum_i v_i(a^*)$

If the values are again, unknown, we can define a game such that:

- a strategy for each player is a function $b_i : A \to \mathbb{R}$

- player i reports $b_i(.)$
- picks a^* that maximizes $\sum_i b_i(a^*)$ (in other words, treat the bids as if they were the values)
- charge player i with

$$p_i = \left[\max_{a \in A} \sum_{j \neq i} b_j(a)\right] - \sum_{j \neq i} b_j(a^*)$$

as we will see, the previous game solves the problem we are trying to solve, because it is truthful (and therefore, each player will report bid $b_i(a) = v_i(a)$).

VCG Mechanism is truthful

Theorem 1. VCG is truthful (in other words: $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all b_i).

Proof.

$$\begin{aligned} u_i(b_i, b_{-i}) &= v_i(a^*) - p_i \\ &= v_i(a^*) - \left[\left[\max_{a \in A} \sum_{j \neq i} b_j(a)\right] - \sum_{j \neq i} b_j(a^*)\right] \\ &= \underbrace{\left[v_i(a^*) + \sum_{j \neq i} b_j(a^*)\right]}_{\text{depends on } b_i \text{ through } a^*} - \underbrace{\left[\max_{a \in A} \sum_{j \neq i} b_j(a)\right]}_{\text{doesn't depend on } b_i} \end{aligned}$$

Remember that a^* maximizes $b_i(a^*) + \sum_{j \neq i} b_j(a^*)$, and player i wants to maximize $v_i(a^*) + \sum_{j \neq i} b_j(a^*)$. So its best strategy is to bid his actual value.

Properties of the VCG Mechanism

There are (at least) two interesting properties of the VCG mechanism.

The first one is that $p_i \geq 0$ (in other words, the auctioneer never pays to the players). This is clear by the definition of p_i. This property is called **no-positive transfers**.

The second property is that (if $v_i \geq 0$ then) $u_i \geq 0$ (i.e., the players "play because they want"). To see this:

$$u_i(v_i, b_{-i}) = \max_{a^* \in A} \left[v_i(a^*) + \sum_{j \neq i} b_j(a^*)\right] - \max_{a \in A} \sum_{j \neq i} b_j(a) \geq 0$$

Example: Single Item Auction

Here the alternatives are $A = \{1, 2, \ldots, n\}$, the player we choose to win. If $\tilde{v}_i \in \mathbb{R}$ is the value of that item for each player then $v_i : A \to \mathbb{R}$ is $v_i(i) = \tilde{v}_i$ and $v_i(j) = 0$.

The alternative selected $i^* \in A$ is the one which maximizes $\sum_i b_i(i^*)$ $(= \max_i \tilde{v}_i$ if truthful).

The player each player pays is $p_i = \left[\max_{a \in A} \sum_{j \neq i} b_j(a)\right] - \sum_{j \neq i} b_j(i^*)$. If $i = i^*$ then the second term is 0 and the first term is the second highest bid. If $i \neq i^*$ then both the first and the second term are $\tilde{v}_{i^*}$ and $p_i = 0$.

Example: Multiple Item Auction

Here we have the following setting:

- n players
- n houses
- player i has a value $\tilde{v}_{ij}$ for house J

The set of alternatives is $A = \{\text{all matchings from players to houses}\}$.

If we had all the values, maximizing the social welfare is the problem of finding a matching of maximum total value, which is known as the weighted bipartite matching problem.

In this case, we ask for bids $\tilde{b}_{ik}$ (we will think of this as the function $b_i(a) = \tilde{b}_{ik}$ if in the matching a, the house k goes to player i. To select the alternative (matching) a^* we solve a maximum weighted bipartite matching problem using the bids as weights. Let a_i be the house given to player i under alternative a. The price will be:

$$p_i = \left[\max_{a \in A} \sum_{j \neq i} b_j(a)\right] - \sum_{j \neq i} b_j(a^*) = \left[\max_{a \in A} \sum_{j \neq i} \tilde{b}_{ja_j}\right] - \sum_{j \neq i} \tilde{b}_{ja_j^*}$$

The second part of the equation above is a simple computation, and the first part is simply a maximum weighted bipartite matching where we set to 0 all the weights of player i to all houses.

Another interpretation to this is that player i should pay "the harm" it causes to the other players (the difference from the benefit they would have without him and how much they have with him).