Lecture 18 – Monday 23 January 2012 - The VCG Mechanism

Overview/Review

The purpose of this lecture is to show the VCG mechanism, which generalizes the Vickery Auction to much more general settings. It does this by giving a pricing mechanism so that to make the auction truthful (i.e., each player’s best strategy is to bid his true value).

Remember our definition of a single item auction:

- $n$ players
- a value $v_i$ for each player $i$
- goal: pick a winner $i^*$ maximizing the social welfare $SW(i^*) = v_{i^*}$
- each player maximizes each own utility: $u_i = v_i - p$ if $i = i^*$ and $u_i = 0$ otherwise.

In this setting, we have a truthful auction: the Second Price Auction (or Vickery Auction):

- each player $i$ submits a bid $b_i$ to the auctioneer
- the player with the highest bid wins the auction ($i^* = \text{Arg max } b_i$)
- player $i^*$ pays the second highest bid

This is interesting because we had an optimization problem (find the maximum $v_i$) over unknown input. So we defined a game to solve it.

Vickery-Clarke Groves mechanism

We are interested in solving the following optimization problem:

- $n$ players
- a set of alternatives $A$ that we can perform
- player $i$ has a value $v_i(a)$ for each $a \in A$
- If alternative $a^*$ is selected, player $i$’s utility is $u_i = v_i(a^*) - p_i$
- goal is to select the alternative $A^*$ that maximizes the social welfare: $\sum_i v_i(a^*)$

If the values are again, unknown, we can define a game such that:

- a strategy for each player is a function $b_i : A \rightarrow \mathbb{R}$
• player $i$ reports $b_i(.)$
• picks $a^*$ that maximizes $\sum_i b_i(a^*)$ (in other words, treat the bids as if they were the values)
• charge player $i$ with

$$p_i = \left[ \max_{a \in A} \sum_{j \neq i} b_j(a) \right] - \sum_{j \neq i} b_j(a^*)$$

as we will see, the previous game solves the problem we are trying to solve, because it is truthful (and therefore, each player will report bid $b_i(a) = v_i(a)$).

**VCG Mechanism is truthful**

**Theorem 1.** VCG is truthful (in other words: $u_i(v_i, b_{-i}) \geq u_i(b_i, b_{-i})$ for all $b_i$).

**Proof.**

$$u_i(b_i, b_{-i}) = v_i(a^*) - p_i$$

$$= v_i(a^*) - \left[ \max_{a \in A} \sum_{j \neq i} b_j(a) \right] - \sum_{j \neq i} b_j(a^*)$$

$$= \left[ v_i(a^*) + \sum_{j \neq i} b_j(a^*) \right] - \max_{a \in A} \sum_{j \neq i} b_j(a)$$

depends on $b_i$ through $a^*$ doesn’t depend on $b_i$

Remember that $a^*$ maximizes $b_i(a^*) + \sum_{j \neq i} b_j(a^*)$, and player $i$ wants to maximize $v_i(a^*) + \sum_{j \neq i} b_j(a^*)$. So its best strategy is to bid his actual value.

**Properties of the VCG Mechanism**

There are (at least) two interesting properties of the VCG mechanism.

The first one is that $p_i \geq 0$ (in other words, the auctioneer never pays to the players). This is clear by the definition of $p_i$. This property is called no-positive transfers.

The second property is that (if $v_i \geq 0$ then) $u_i \geq 0$ (i.e., the players “play because they want”).

To see this:

$$u_i(v_i, b_{-i}) = \max_{a^* \in A} \left[ v_i(a^*) + \sum_{j \neq i} b_j(a^*) \right] - \max_{a \in A} \sum_{j \neq i} b_j(a) \geq 0$$
Example: Single Item Auction

Here the alternatives are $A = \{1, 2, \ldots, n\}$, the player we choose to win. If $\tilde{v}_i \in \mathbb{R}$ is the value of that item for each player then $v_i : A \rightarrow \mathbb{R}$ is $v_i(i) = \tilde{v}_i$ and $v_i(j) = 0$.

The alternative selected $i^* \in A$ is the one which maximizes $\sum_i b_i(i^*)$ (= $\max_i \tilde{v}_i$ if truthful).

The player each player pays is $p_i = \left[ \max_{a \in A} \sum_{j \neq i} b_j(a) \right] - \sum_{j \neq i} b_j(i^*)$. If $i = i^*$ then the second term is 0 and the first term is the second highest bid. If $i \neq i^*$ then both the first and the second term are $\tilde{v}_{i^*}$ and $p_i = 0$.

Example: Multiple Item Auction

Here we have the following setting:

- $n$ players
- $n$ houses
- player $i$ has a value $\tilde{v}_{ij}$ for house $J$

The set of alternatives is $A = \{\text{all matchings from players to houses}\}$.

If we had all the values, maximizing the social welfare is the problem of finding a matching of maximum total value, which is known as the weighted bipartite matching problem.

In this case, we ask for bids $\tilde{b}_{ik}$ (we will think of this as the function $b_i(a) = \tilde{b}_{ik}$ if in the matching $a$, the house $k$ goes to player $i$). To select the alternative (matching) $a^*$ we solve a maximum weighted bipartite matching problem using the bids as weights. Let $a_i$ be the house given to player $i$ under alternative $a$. The price will be:

$$p_i = \left[ \max_{a \in A} \sum_{j \neq i} b_j(a) \right] - \sum_{j \neq i} b_j(a^*) = \left[ \max_{a \in A} \sum_{j \neq i} \tilde{b}_{ja_j} \right] - \sum_{j \neq i} \tilde{b}_{ja_j^*}$$

The second part of the equation above is a simple computation, and the first part is simply a maximum weighted bipartite matching where we set to 0 all the weights of player $i$ to all houses.

Another interpretation to this is that player $i$ should pay “the harm” it causes to the other players (the difference from the benefit they would have without him and how much they have with him).