In this lecture, we mainly discussed how to generalize our previous results in single item auction on matroids.

1 Matroid

First we will give the definition of a matroid. A matroid is a set system \((E,I)\) where \(E\) is the ground set of elements and \(I\) is a set of independent subsets of \(E\).

**Definition.** Independent sets satisfies:

- if \(A\) is independent, \(B \subset A\), then \(B\) is also independent.
- if \(A\) and \(B\) are both independent, \(|A| > |B|\), then \(\exists a \in A \setminus B\), so that \(B \cup \{a\}\) is independent.

An immediate corollary is that independent sets with maximum size are all with the same size. It is called *rank*.

In the context of auction, independence means that we can sell a service to each element. In an auction, each person \(s\) is with value \(v_s\), we sell service to subset \(A\) at some price, \(A\) must be independent.

Here are some examples.

- Selling single item, the corresponding matroid: 1-element sets and \(\emptyset\)
- Selling \(k\) items, the corresponding matroid: \(\leq k\)-element sets
- Each player \(s\) with a vector \(z_s \in \mathbb{R}^d\), the corresponding matroid: linear independence, \(A\) s.t. \(\{z_s : s \in A\text{linear independent}\}\)
- matchable sets, in a bipartite graph, each client can only buy items connected by an edge, the corresponding matroid: valid matchings.

Now recall the results in single item auction,
(a) sell service in matroid structure, players’ values are drawn from identical regular distribution \( F \):
- maximize social welfare
- maximize revenue

(b) (a) with non-regular distribution.

(c) 2-approximation if distribution of \( v_i \) is from a regular distribution \( \hat{F}_i \).

(d) ???-approximation if distribution of \( v_i \) is from a non-regular distribution \( \hat{F}_i \).

In this lecture, we covered (a) and (b).

2 Generalization on matroids

2.1 Maximizing social welfare

In order to maximize social welfare \( \sum_{i \in I} v_i \), select independent set \( I \) with Algorithm 1.

1) Ask value \( v_i \), sort \( v_1 \geq \ldots \geq v_n \);
2) \( I = \emptyset \);
3) for \( i = 1, \ldots, n \) do
4) \quad add \( v_i \) to \( I \) if \( I + \{i\} \) is independent and \( v_i \geq 0 \);
5) end

Algorithm 1: Greedy Algorithm

**Theorem 1.** Greedy algorithm maximizes \( \sum_{v \in I} v_i \) over matroids.

**Proof.** The proof is from “Approximation in Economic Design” by Jason Hartline.

Let \( r \) be the rank of the matroid. Let \( I = \{i_1, \ldots, i_r\} \) be the set of elements selected in the surplus maximizing assignment, and let \( J = \{j_1, \ldots, j_r\} \) be the set of elements selected by the greedy algorithm. The surplus from serving a subset \( S \) of the agents is \( \sum_{i \in S} v_i \). Assume for a contradiction that the surplus of set \( I \) is strictly more than the surplus of set \( J \), i.e., greedy algorithm is not optimal. Assume the items of \( I \) and \( J \) are indexed in decreasing order. Therefore, there must exist a first index \( k \) such that \( v_{i_k} > v_{j_k} \). Let \( I_k = \{i_1, \ldots, i_k\} \) and let \( J_{k-1} = \{j_1, \ldots, j_{k-1}\} \). Applying the augmentation property to sets \( I_k \) and \( J_{k-1} \) we see that there must exist some element \( i \in I_k \setminus J_{k-1} \) such that \( J_{k-1} \cup \{i\} \) is feasible. Of course, \( v_i \geq v_{i_k} > v_{j_k} \) which means agent \( i \) was considered by greedy algorithm before it selected \( j_k \). According to the definition of independent sets, \( J_{k-1} \cup \{i\} \), when \( i \) was considered by greedy algorithm and it was feasible. By definition of the algorithm, \( i \) should have been added; this is a contradiction.

This is VCG. Ask players about values, find the solution that maximizes social welfare and use price characterization to set price.

Note in this greedy algorithm, no assumption is needed.
2.2 Maximizing expected revenue

The goal is to maximize $E[\sum_{i \in I} \phi_i(v_i)]$, $\phi_i(v_i)$ is the virtual value of player $i$ when value is $v_i$.

If distributions are identical, $\phi_i(v) = \phi(v)$, the goal becomes maximizing $E(\sum_{i \in I} \phi(v_i))$.

If the distribution is regular, then $\phi(v)$ is monotone non-decreasing. If $v_1 \geq \ldots \geq v_n$, then $\phi(v_1) \geq \ldots \geq \phi(v_n)$.

We can just run the same greedy algorithm 1 with reserve price $r$ such that $\phi(r) = 0$.

2.3 Non-regular distribution

As what we did in single item auction, we need to consider ironed virtual value $\bar{\phi}(q) = \bar{R}'(q)$, where $\bar{R}(q)$ is the smallest concave function no smaller than the revenue with sale prob $q$, $R(q) = qv(q)$.

$\bar{\phi}(q)$ is monotone non-increasing in $q$, monotone non-decreasing in value $v$.

As we have shown in last lecture, the goal is to maximize revenue $E(\sum_{i \in I} \bar{\phi}(v_i))$.

Note if $v_1 \geq \ldots \geq v_n$, then $\bar{\phi}(v_1) \geq \ldots \geq \bar{\phi}(v_n)$, set reserve price at $\bar{\phi}(r) = 0$. One important property we needed last time is that in regions where $\bar{R}_i(q) > R_i(q)$ allocation $x_i$ is fixed. (If true, $E(\sum \bar{\phi}_i(v_i)) = E(\sum \phi_i(v_i))$. In this region $\bar{\phi}_i(v)$ is constant.

Then the algorithm needs changing a little:

Sort by $\bar{\phi}$, choose between identical $\bar{\phi}$ values by some consistent rules not using the order of values, for example random or alphabetically.