1 Price of Anarchy for non-atomic congestion game

**Theorem 1.** If the delay functions are $(\lambda, \mu)$-smooth (for all $x, y \ yd(x) \leq \lambda yd(y) + \mu xd(x)$), then the total delay in a Nash equilibrium $\leq \frac{1}{1-\mu}$ total delay in optimum, where total delay is equal to $
abla \sum_{p} fp(\sum_{e \in p} de(x)) = \sum_{e} xe de(x)$.

**Proof.** Let $f$ be the flow at a Nash equilibrium and $X$ be the congestion it creates, and let $f^*$ be the flow at optimum and $X^*$ be the congestion it creates. Let $\delta_1, \ldots, \delta_N$ be disjoint groups of $r_1, \ldots, r_n$, such that all members of $\delta_i$ are of the same type and all use $P_i$ in $f$ and $P_i^*$ in $f^*$.

We know that for each member of $\delta_i$,

\[
\sum_{e \in P_i} de(x) \leq \sum_{e \in P_i^*} de(x)
\]

We can multiply this by $\delta_i$ and sum for all $i$ and we get

\[
\sum_i \delta_i \sum_{e \in P_i} de(x) \leq \sum_i \delta_i \sum_{e \in P_i^*} de(x)
\]

Changing the order of summation we get

\[
\sum_{e} de(x) \sum_{i \in P_i} \delta_i \leq \sum_{e} de(x) \sum_{i \in P_i^*} \delta_i
\]

We now notice that $\sum_{i \in P_i} \delta_i = x_e$ and $\sum_{i \in P_i^*} \delta_i = x_e^*$, So by using smoothness we get

\[
\sum_{e} de(x) x_e \leq \sum_{e} de(x) x_e^* \leq \lambda \sum_{e} de(x) x_e^* + \mu \sum_{e} de(x) x_e
\]

Rearranging terms we get what we wanted to prove. $\square$

2 Price of Anarchy for the discrete version

We use a more general game formalization:

- $n$ players, numbered $1 \ldots n$
- each player has a strategy set $S_i$
- Given a strategy $s_i \in S_i$ for each player, each player has a cost function, $C_i(S)$, which is a function of the strategy vector $S = (s_1 \ldots s_n)$
- we say that $S = (s_1 \ldots s_n)$ is a Nash equilibrium if for every player $i$ and for every strategy $s'_i \in S_i$, $C_i(S) \leq C_i(s'_i, S_{-i})$ ($S_{-i}$ is the vector where all coordinates except for $i$ are the same as in $S$).
• We say that such a game is \((\lambda, \mu)\)-smooth if for all strategy vectors \(S, S^*\) \[ \sum_i C_i(S^*_i, S_{-i}) \leq \lambda \sum_i C_i(S^*) + \mu \sum_i C_i(S). \]

**Theorem 2.** (Roughgarden ’09) If a game is \((\lambda, \mu)\)-smooth for \(\mu < 1\) then the total cost at a Nash equilibrium is \(\leq \frac{\lambda}{1-\mu}\) minimum possible total cost.

**Proof.** Let \(S\) be the strategy vector in a Nash equilibrium and \(S^*\) be a strategy vector in a minimum cost solution. From Nash we know that
\[ C_i(S) \leq C_i(S^*_i, S_{-i}) \]

We can sum this for all i’s, apply smoothness and get
\[ \sum_i C_i(S) \leq \sum_i C_i(S^*_i, S_{-i}) \leq \lambda \sum_i C_i(S^*) + \mu \sum_i C_i(S) \]

Rearranging terms we get what we wanted to prove. \(\square\)

This gives a general framework for Price of Anarchy proofs, and it was shown that many of the proofs were actually reproving this theorem with specific parameters that matched their settings.

### 2.1 Smoothness for discrete congestion games

Let \(p_1, \ldots, p_n\) and \(p^*_1, \ldots, p^*_n\) be two series of paths chosen by the players that result in congestions \(X\) and \(X^*\).

We say that a discrete congestion game is \((\lambda, \mu)\)-smooth if for all such \(P\) and \(P^*\),
\[ \sum_{e \in P \cap p_i} d_e(x_e) + \sum_{e \in P \setminus p_i} d_e(x_e + 1) \leq \lambda \sum_{e} x^*_{e} d_e(x^*_e) + \mu \sum_{e} x_{e} d_e(x_e). \]