CS 6840 – Algorithmic Game Theory (2 pages) Spring 2012

Lecture 5 Scribe Notes

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1 Price of Anarchy for non-atomic congestion game

Theorem 1. If the delay functions are (λ, μ)-smooth (for all x, y $yd(x) \leq \lambda yd(y) + \mu xd(x)$), then the total delay in a Nash equilibrium $\leq \frac{\lambda}{1-\mu}$ total delay in optimum, where total delay is equal to $\sum_P f_P(\sum_{e \in P} d_e(x_e)) = \sum_e x_e d_e(x_e)$.

Proof. Let f be the flow at a Nash equilibrium and X be the congestion it creates, and let f^* be the flow at optimum and X^* be the congestion it creates. Let $\delta_1, \dots, \delta_N$ be disjoint groups of $r_1, \dots, r_n$, such that all members of δ_i are of the same type and all use P_i in f and P_i^* in f^*.

We know that for each member of δ_i

$$\sum_{e \in P_i} d_e(x_e) \leq \sum_{e \in P_i^*} d_e(x_e)$$

We can multiply this by δ_i and sum for all i and we get

$$\sum_i \delta_i \sum_{e \in P_i} d_e(x_e) \leq \sum_i \delta_i \sum_{e \in P_i^*} d_e(x_e)$$

Changing the order of summation we get

$$\sum_e d_e(x_e) \sum_{i: e \in P_i} \delta_i \leq \sum_e d_e(x_e) \sum_{i: e \in P_i^*} \delta_i$$

We now notice that $\sum_{i: e \in P_i} \delta_i = x_e$ and $\sum_{i: e \in P_i^*} \delta_i = x_e^*$. So by using smoothness we get

$$\sum_e d_e(x_e) x_e \leq \sum_e d_e(x_e) x_e^* \leq \lambda \sum_e d_e(x_e^*) x_e^* + \mu \sum_e d_e(x_e) x_e$$

Rearranging terms we get what we wanted to prove. □

2 Price of Anarchy for the discrete version

We use a more general game formalization:

- n players, numbered $1 \dots n$
- each player has a strategy set S_i
- Given a strategy $s_i \in S_i$ for each player, each player has a cost function, $C_i(S)$, which is a function of the strategy vector $S = (s_1 \dots s_n)$
- we say that $S = (s_1 \dots s_n)$ is a Nash equilibrium if for every player i and for every strategy $s_i' \in S_i$, $C_i(S) \leq C_i(s_i', S_{-i})$ (S_{-i} is the vector where all coordinates except for i are the same as in S).

- We say that such a game is (λ, μ)-smooth if for all strategy vectors S, S^* $\sum_i C_i(S_i^*, S_{-i}) \leq \lambda \sum_i C_i(S^*) + \mu \sum_i C_i(S)$.

Theorem 2. (Roughgarden '09) If a game is (λ, μ)-smooth for $\mu < 1$ then the total cost at a Nash equilibrium is $\leq \frac{\lambda}{1-\mu}$ minimum possible total cost.

Proof. Let S be the strategy vector in a Nash equilibrium and S^* be a strategy vector in a minimum cost solution. From Nash we know that

$$C_i(S) \leq C_i(S_i^*, S_{-i})$$

We can sum this for all i's, apply smoothness and get

$$\sum_i C_i(S) \leq \sum_i C_i(S_i^*, S_{-i}) \leq \lambda \sum_i C_i(S^*) + \mu \sum_i C_i(S)$$

Rearranging terms we get what we wanted to prove. □

This gives a general framework for Price of Anarchy proofs, and it was shown that many of the proofs were actually reproving this theorem with specific parameters that matched their settings.

2.1 Smoothness for discrete congestion games

Let $p_1, \ldots, p_n$ and $p_1^*, \ldots, p_n^*$ be two series of paths chosen by the players that result in congestions X and X^*.

We say that a discrete congestion game is (λ, μ)-smooth if for all such P and P^*, $\sum_i (\sum_{e \in p_i^* \cap p_i} d_e(x_e) + \sum_{e \in p_i^* \setminus p_i} d_e(x_e + 1)) \leq \lambda \sum_e x_e^* d_e(x_e^*) + \mu \sum_e x_e d_e(x_e)$.