1 Lecture 38 – Friday 20 April 2012 - Price Equilibrium in Arrow-Debreu Model

1.1 Setup
- Goods \{1, ..., k\}.
- Players \{1, ..., n\}.
- Player i brings \(\bar{w}_i = (w_1, ..., w_k)\) amount of goods to the market, and has utility \(U_i(\bar{x}_i)\), where \(\bar{x}_i = (x_{i1}, ..., x_{ik})\), where \(x_{ij}\) = amount of good j that player i gets.
- Assume utilities \(U_i(\cdot)\) strictly monotone increasing, strictly concave, continuously differentiable.

1.2 Price Equilibrium
Let \(p = (p_1, ..., p_k)\) be the prices for each good. Each player i sells \(\bar{w}_i\) to get \(p \cdot \bar{w}_i\) amount of money that is used for trading. Given prices, each player finds
\[
\bar{x}_i = \arg\max_{\bar{x}} \{U_i(\bar{x}) : p \cdot \bar{x} \leq p \cdot \bar{w}_i, \bar{x} \geq 0\}
\]
Note that since \(U_i(\cdot)\) is strictly concave, \(\bar{x}_i\) is unique. Also, since \(U_i(\cdot)\) is strictly monotone increasing (in every dimension), \(p \cdot \bar{x}_i = p \cdot \bar{w}_i\).

**Definition.** Prices \(p = (p_1, ..., p_k), p_j > 0\) is a price equilibrium if the resulting \(\bar{x}_1, ..., \bar{x}_n\) optima satisfy:
\[
\forall j \sum_i x_{ij} \leq \sum_i w_{ij}
\]
Note that by strict monotonicity of utilities, if \(p_j = 0\) then all users want \(x_{ij} = \infty\), so that cannot be an equilibrium.

**Lemma** (Market clearing). For all goods \(j\), \(\sum_i x_{ij} = \sum_i w_{ij}\).

**Proof.** As noted earlier, we have
\[
\sum_i p \cdot \bar{x}_i = \sum_i p \cdot \bar{w}_i
\]
\[
\sum_j p_j \sum_i x_{ij} = \sum_j p_j \sum_i w_{ij}
\]
The only way for this to be equal is that they are term-by-term equal, so
\[ p_j \sum_i x_{ij} = p_j \sum_i w_{ij} \]
\[ \sum_i x_{ij} = \sum_i w_{ij} \]

More generally, \( p_j (\sum_i x_{ij}) - \sum_i w_{ij} \) = 0 even if \( U_i(\cdot) \) is only monotone increasing.

**Definition** (Simplex). \( \Delta_n := \{ x \in \mathbb{R}^n : x_i \geq 0, \sum_i x_i = 1 \} \).

**Theorem 1** (Brouwer Fixed Point Theorem). If function \( f : \Delta_n \to \Delta_n \) is continuous, then there exists \( x \) such that \( f(x) = x \).

**Theorem 2.** Equilibrium prices exist.

**Proof.** Note that if \( p \) is a price equilibrium, then \( \alpha p \) is also a price equilibrium for any \( \alpha > 0 \). WLOG, restrict to prices such that \( p \in \Delta_n \). Let \( \bar{x}_1, ..., \bar{x}_n \) be user optima, and let

\[ e_j = \left[ \sum_j (x_{ij} - w_{ij}) \right]^+ \]
\[ f(p) = \bar{p} \]
\[ \forall j \quad p_j = \frac{p_j + e_j}{\sum_i (p_i + e_i)} \]

**Lemma 3.** \( p \) is price equilibrium \( \iff f(p) = p \).

**Proof.** Clearly, \( p \) is price equilibrium \( \implies f(p) = p \). Thus, we only need to show that if \( p \) is not a price equilibrium, then \( p \) is not a fixed point of \( f \). Note that price changes unless \( e_j/p_j \) is fixed for all \( j \). We claim that there exist a good \( j \) such that \( \sum_i x_{ij} \leq \sum_i w_{ij} \). Recall,
\[ \sum_j p_j \sum_i x_{ij} = \sum_j p_j \sum_i w_{ij} \]
Hence, it cannot be the case that \( e_j > 0 \) and for all goods \( j \), \( \sum_i x_{ij} > \sum_i w_{ij} \). Thus, if \( p \) is not a price equilibrium, then there is some good \( j \) such that \( e_j > 0 \) and hence, there must be some good that will have its price reduced under \( f \), so \( p \) is not a fixed point of \( f \).

**Lemma 4.** \( f \) is continuous.

**Proof.** \( \bar{p} \) is continuous, and \( e_j \) is continuous, so we only need \( \bar{x}_i \) to be continuous for all players \( i \). Using a fact from continuous optimization, optimizer \( \bar{x}_i \) (unique) is a continuous function of \( p \), so \( f \) is continuous.

**Lemma 5.** \( f : \Delta_n \to \Delta_n \) is a function. If prices are zero, then \( x_{ij} = \infty \) and \( e_j \) is unbounded. Hence, we need \( \bar{x}_i \)'s to be bounded to make \( e_j \)'s bounded. To do this, we modify the user optimization to include an extra condition.
\[ \bar{x}_i = \arg \max_{\bar{x}} \left\{ U_i(\bar{x}) : p\bar{x} \leq p\bar{w}, \quad \forall j, x_j \geq 0, \quad \forall j, x_j \leq \sum_i w_{ij} + 1 \right\} \]
Note that the last condition cannot be tight at the fixed point as it violates price equilibrium conditions. Hence, this does not change the problem, but ensures that \( \bar{x}_i \)'s are bounded, and \( f : \Delta_n \rightarrow \Delta_n \) is indeed a function.

Applying Brouwer’s fixed point theorem to \( f \) shows that price equilibrium \( p \) exists.