CS 6840 Notes

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Bayesian Auctions

Last time - single-item auction

- User's value drawn independently from distribution $\mathcal{F}$.
- There are n users, what we know about them is the all the same.

Two types of auctions

- Second price auction (select $\max_i b_i$)
- Fixed price p such that $Pr(v > p) = \frac{1}{n}$

Notation

- User has value v_i (drawn from $\mathcal{F}$)
- Social welfare is v_i for i that receives the item.
  This could be in expectation if we randomize who gets the item.

Today - First Price Auction

This is a traditional game, in contrast to the second price auction where it is optimal to bid truthfully.

- Distribution $\mathcal{F}$ is known, player values v_i are drawn independently, and all players know this distribution
- Player bids b_i
- Select $\max_i b_i$, and maximum bidder gets item and pays b_i
- Benefit is $v_i - p_i$ if player i wins

Bidding $b_i = v_i$ guarantees no gain If you don't win, you gain nothing. If you win, then the net value of what you gain is still 0.

Theorem

The following bid is a Nash equilibrium:

$$b(v) = \text{bid if value is } v = Ex(\max_{j \neq i} v_j \mid v_j \leq v \ \forall\, j)$$

Conditioning: v is highest. Expectation: expected value of 2nd highest.

In other words, what's the expected value of the second highest bid, supposing that you have the highest bid?

Assuming all players use this bidding strategy, does the player with the highest value win? Deterministically yes! Each $b_i(v)$ is the same function (independent of i). Also, $b(v)$ is monotone in v, so the highest value wins. This results in the same outcome as the second price auction (=outcome-equivalent to the second price auction).

Is this also revenue-equivalent to the second price auction? Yes! Suppose you are a player i with value v. You are winning with the same probability as in a second-price auction. In fact, for each player, price is same as the expected price in a second-price auction.

Proof of Theorem

Suppose player 1 feels like deviating. Suppose that his bid $b_1(v)$ has a range from 0 to some unknown number.

Is it better to bid above any other player's range? No. Since you're guaranteed to win by bidding just at the top of someone's range anyway.

So consider a plausible alternate bid $b(z) < v$. This is effectively bluffing that your value is z rather than v.

Goal: solve calculus problem of what the best z is. If $z = v$, then $b()$ is Nash.

Player with $b(z)$ if $z \geq v_i \ \forall \ i > 1$ then the probability is $Pr(\max_{i>1} v_i < z)$.

What you pay is $b(z) = Ex(\max_{i>1} v_i | v_i < z \ \forall \ i)$

Thus, your expected value is

$$Pr(\max_{i>1} v_i < z) \left(v - Ex(\max_{i>1} v_i | v_i < z \ \forall \ i) \right)$$

Let $\max_{i>1} v_i = X$, a random variable.

Rewritten, the expected value is

$$Pr(X < z) \cdot (v - E(X | X \leq z))$$

The expected value can be written as

$$Pr(X < z)v - \int_0^\infty (Pr(X < z) - Pr(X < \xi))d\xi$$
$$= Pr(X < z)v - Pr(X < z)z + \int_0^z Pr(X < \xi)d\xi$$

Taking derivatives w.r.t z, we get

$$-Pr(X < z) + Pr(X < z) + Pr(X < z)'(v - z)$$

by the product rule and because the derivative of an integral is the value inside, and simplifying,

$$Pr(X < z)'(v - z)$$

To maximize our expectation, set $v = z$, and we need to verify that this is a maximum by checking that $Pr(X < z)$ is monotone and hence its derivative is positive.

Side Note about Expectations and Probabilities

X is any random variable ≥ 0.

$Ex(X) = \int_0^\infty (1 - Pr(X < z))dz = \int_0^\infty Pr(X \geq z)dz$

Why? If X is discrete, taking values $1, ..., u$, $E(X) = \sum_i i \cdot Pr(X = i) = \sum_i Pr(X \geq i)$. The continuous version of this follows immediately.

Also,

$Ex(X|X < z) \cdot Pr(X < z) = Ex(X^z)$ where $X^z = \begin{cases} X \text{ if } X < z \\ 0 \text{ otherwise} \end{cases}$

$= \int_0^\infty Pr(X < z) - Pr(X < \xi)d\xi = \int_0^\infty Pr(z > X > \xi)d\xi$