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Lecture 3: Continuous Congestion Games

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1 Review: Atomic Congestion Games

Recall the definition of an Atomic Congestion Game from last lecture, which consisted of the following:

- E, a finite set of congestible elements.
- Players $i \in \{1, \ldots, n\}$, each with a strategy set S_i, where each strategy $P \in S_i$ is a subset of E. (Each strategy choice "congests" some of the congestible elements.)
- Delay functions $d_e \geq 0$ for each $e \in E$.

Further, given a set of strategy choices $P_i \in S_i$ for each player i, we defined the following:

- The *congestion* on an element e, given by $x_e = |\{i : e \in P_i\}|$, the number of players congesting that element.
- The *delay* on each element e, given by $d_e(x_e)$.
- The *cost* for each player i, equal to $\sum_{e \in P_i} d_e(x_e)$, the sum of delays for all elements used by that player.

We also defined a set of strategies to be a *Nash Equilibrium* if no single player could improve their cost by swapping only his/her own strategy. More formally,

$$\forall i, \forall Q_i \in S_i, \sum_{e \in P_i} d_e(x_e) \leq \sum_{e \in P_i \cap Q_i} d_e(x_e) + \sum_{e \in Q_i - P_i} d_e(x_e + 1)$$

We also showed that each Atomic Congestion Game has a Nash Equilibrium, and in fact this Nash Equilibrium can be found quite naturally, by performing *iterative best response*. Our proof of used the following potential function:

$$\Phi = \sum_{e \in E} \sum_{1}^{x_e} d_e(x_e)$$

We showed that each step of the iterative best response algorithm strictly reduced the value of this potential function, with the decrease in Φ being exactly the decrease in the cost of the user changing his/her strategy in that iteration. Further, we showed that any local minimum corresponds to a Nash Equilibrium.

We noted an inelegance of the atomic version of congestion games was that the expression for Nash Equilibria contains a "+1". When the number of players is very large, this 1 player should have a very tiny effect. With this in mind, we defined a non-atomic version of Congestion Games.

2 Non-Atomic Congestion Games

Our definition of non-atomic congestion games uses the fact that players are now infinitesimally small. We have the following components:

- The finite set of congestible elements E, which remains the same.
- Instead of n players, we have n *types* of players, with a number r_i reflecting the "amount" of players of type i. Each type i selects from strategy set S_i, and we assume for simplicity that the S_i are mutually disjoint. (The r_i can be thought of as "rate" of traffic between a particular source and sink, for example).
- The delay functions, d_e for every $e \in E$, are now assumed to be continuous.
- We allow each type of players to distribute fractionally over their strategy set. We let $f_P \geq 0$ represent the amount of players using strategy P. Then we have the constraint $\sum_{P \in S_i} f_P = r_i$, that is, all the players of type i have some strategy.
- The congestion on e is defined similarly to the atomic case: $x_e = \sum_{P: e \in P} f_P$.

A choice of strategies, f_P, is now said to be a Nash Equilibrium if the following holds:

$$\forall i, \forall P \in S_i \text{ s.t. } f_P > 0, \forall Q \in S_i, \sum_{e \in P} d_e x_e \leq \sum_{e \in Q} d_e(x_e)$$

The equation reflects the fact that not even changing the strategy of a tiny amount of players of a single type can decrease the cost experienced.

We now wish to show that such a Nash Equilibrium exists.

3 Existence of a Nash Equilibrium

We will utilize the non-atomic analogue of the potential function from atomic games:

$$\Phi = \sum_{e \in E} \int_o^{x_e} d_e(z) dz$$

We claim that the minimum of this function is a Nash Equilibrium. But how do we know that such a minimum exists?

Observe first that Φ is continuous. This follows from the fact that the inner terms are integrals of a continuous function (d_e) with a continuous upper limit, and are thus contínous. Further, the sum of continuous functions is also continuous. It follows that Φ is continuous.

Also notice that the set we are optimizing over is *compact*, and continuous functions have minima over compact sets.

[*Note* : A compact set is one that is bounded and contains the limit of every convergent sequence of elements the set. For example, $[0, \infty)$ is not compact because it is not bounded, and $[0, 2)$ is not compact because we can construct an infinite sequence of numbers converging to 2, but 2 is not in the set.

Note also that any decreasing function, for example $f(x) = 7 - 2x$, does not have a minimum over these sets, because we can always find an element with smaller value.

Further, the set we are optimizing over is bounded from above and below, because we have the restrictions that $f_P \geq 0$ and $\sum_{P \in S_i} f_P = r_i$, and this set is also "closed", that is, it contains the limits of all sequences in it. Hence it is a compact set.]

It follows that there exists a set of f_P minimizing Φ. It remains to show that a minimum of Φ is actually a Nash Equilibrium.

Claim 1 *A minimum of Φ is a Nash Equilibrium.*

Proof. We give a somewhat informal proof of this claim.

Suppose that we have a set of f_P minimizing Φ that is not a Nash equilibrium. Then $\exists i, P \in S_i$ with $f_p > 0$ and $\exists Q \in S_i$ such that

$$\sum_{e \in P} d_e(x_e) > \sum_{e \in Q} d_e(x_e)$$

The idea is to take a tiny amount $\delta < f_P$ of players using strategy P and change them to strategy Q, that is, change the strategies to $f_P - \delta$ and $f_Q - \delta$.

Notice that increasing f_Q by δ increases x_e for $e \in Q$ by the same δ. This has the effect of increasing changing the x_e term in Φ to $\int_0^{x_e+\delta} d_e(x_e)$. Since d_e is continuous, the change is approximately $\delta \cdot d_e(x_e)$ (with error that is proportional to δ^2, using Taylor bounds from calculus) . A similar argument holds when we decrease f_P by δ.

Then, as long as δ is sufficiently small and the error is sufficiently low (as it is proportional to δ^2), the change in Φ from changing a δ amount of players from P to Q is given by approximately

$$\delta \cdot (\sum_{e \in Q} d_e(x_e) - \sum_{e \in P} d_e(x_e)) < 0$$

Which contradicts the fact that our original set of strategies minimized Φ. It follows that any minimum of Φ must also be a Nash Equilibrium.