CS 6840 – Algorithmic Game Theory — Spring 2012

Lecture 14 – Wednesday 22 February 2012 – Scribe Notes

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Coarse correlated equilibria as a convex set

Last time - Looked at algorithm that guarantees no regret Last last time - Defined coarse correlated equilibrium as a probability distribution on strategy vectors

Definition. $p(s)$ s.t. $E(u_i(s)) \geq E(u_i(x, s_{-i}))\ \forall i, \forall x$.

which lead to the corollary:

Corollary 1. All players using small regret strategies gives an outcome that is close to a coarse correlated equilibrium

The next natural question to ask is: Does there exist a coarse correlated equilibrium? We consider finite player and strategy sets.

Theorem 2. With finite player and strategy sets, a coarse correlated equilibrium exists.

Proof 1. We know that a Nash equilibrium exists. Then let $p_1, \ldots p_n$ be probability distributions that form a Nash equilibrium. Observe that $p(s) = \Pi_i p_i(s_i)$ is a coarse correlated equilibrium. □

Proof 2. (doesn't depend on Nash's theorem). Idea: Algorithm from last lecture finds it with small error. Consider

$$\min_p[\max_i[\max_x[E_p(u_i(x_i, s_{-i})) - E_p(u_i(s))]]]$$

The quantity inside the innermost max is the regret of players i about strategy x. If this minimum is ≤ 0, then p is a coarse correlated equilibrium. The minimum cannot equal $\epsilon > 0$ as we know by the algorithm that we can find a p with arbitrarily small regret. In this instance, $\frac{\epsilon}{2}$ would be sufficient to reach a contradiction. Hence, we know that the infimum must be less than or equal to 0 but does the minimum exist? Since we have a continuous function over p, the compact space of probability distributions, we must attain the infimum, so the minimum is in fact ≤ 0, so a coarse correlated equilibrium exists. □

Remark. This minimum can be calculated as the solution of a linear program satisfying $\sum p(s) = 1, p(s) \geq 0$ and the no regret inequality for each (i, x) pair.

2-person 0-sum games

The game is defined by a matrix a with the first players strategies labelling the rows and the second players strategies labelling the columns. a_{ij} is the amount Player 1 pays to Player 2 if strategy vector (i, j) plays.

Theorem 3. Coarse correlated equilibrium in these games is (essentially) the same as the Nash equilibrium.

To be a bit more precise, let $p(i,j)$ be at coarse correlated equilibrium. When considering Player 1, we care about q Player 2's marginal distribution. $q(j) = \sum_i p(i,j)$. Since Player 1 has no regret, we have that

$$\sum_{ij} a_{ij} p(i,j) \leq \min_i \sum_j a_{ij} q_j$$

Likewise, let $r(i) = \sum_j p(i,j)$ be Player 1's marginal distribution, so Player 2's lack of regret tells that:

$$\sum_{ij} a_{ij} p(i,j) \geq \max_j \sum_i a_{ij} r_i$$

Theorem 4. q, r from above are Nash equilibria.

Proof. The best response to q is

$$\min_i \sum_j a_{ij} q_j \leq \sum_{ij} r(i) q(j) \leq \max_j \sum_i a_{ij} r_i$$

the last of which is the best response to r. Thus, we also have

$$\sum_{ij} a_{ij} p(i,j) \leq \min_i \sum_j a_{ij} q_j \leq \sum_{ij} r(i) q(j) \leq \max_j \sum_i a_{ij} r_i \leq \sum_{ij} a_{ij} p(i,j)$$

Which implies the result, since they must all be equal. □