Lecture Scribe Notes

Instructor: Eva Tardos *Sidharth Telang (sdt45)*

1 Lecture – Friday 17 February 2012 - Other equilibria

The following notation is used. $[n] = \{1, 2, \ldots, n\}$ is used to denote the set of players. Player i has strategy set S_i. $\bar{s}$ denotes a strategy vector. $\bar{s}_i$ denotes the i^{th} entry of $\bar{s}$ and $\bar{s}_{-i}$ denotes $\bar{s}$ without the i^{th} entry. $c_i(\bar{s})$ denotes the cost player i incurs when the players play $\bar{s}$.

We say a sequence of plays $(\bar{s}_1, \bar{s}^2, \ldots \bar{s}^T)$ is no regret for player i if and only if

$$\Sigma_{t=1}^{T} c_i(\bar{s}^t) \leq min_{x \in S_i} \Sigma_{t=1}^{T} c_i(x, \bar{s}_{-i}^t)$$

which means that player i does at least as well as he would have had he chosen any fixed strategy in hindsight.

Recall that a mixed Nash equilibrium is defined as a probability distribution p_i for every player i over S_i such that for every player i and every $x \in S_i$

$$\mathbb{E}(c_i(\bar{s})) \leq \mathbb{E}(c_i(x, \bar{s}_{-i}))$$

where $\bar{s}$ is now a random variable. That is, the probability $\bar{s}$ is played is $\Pi_i p_i(\bar{s}_i)$. Let this be denoted by $p(\bar{s})$.

Here we note that the more natural definition of enforcing the expected cost of any player i under p_i to be no more than that when i switches to any other probability distribution p_i' is equivalent to the above definition. This is because the expected cost of player i on switching to a probability distribution will be a convex combination of his expected cost on switching to fixed strategies.

A sequence of plays defines a probability distribution on the set of strategy vectors. We set $p(\bar{s})$ to be the frequency of $\bar{s}$, that is the number of times $\bar{s}$ was played divided by the total number of plays.

If a sequence of plays are no regret for all players we have for every player i

$$\Sigma_{t=1}^{T} c_i(\bar{s}^t) \leq min_{x \in S_i} \Sigma_{t=1}^{T} c_i(x, \bar{s}_{-i}^t)$$

which is equivalent to the condition that for every player i

$$\Sigma_{\bar{s}} p(\bar{s}) c_i(\bar{s}) \leq min_{x \in S_i} \Sigma_{\bar{s}} p(\bar{s}) c_i(x, \bar{s}_{-i})$$

Such a probability distribution is defined as a coarse correlated equilibrium.

Definition. A coarse correlated equilibrium is defined as a probability distribution p over strategy vectors such that for every player i

$$\Sigma_{\bar{s}} p(\bar{s}) c_i(\bar{s}) \leq min_{x \in S_i} \Sigma_{\bar{s}} p(\bar{s}) c_i(x, \bar{s}_{-i})$$

We have seen that the distribution induced by a sequence of plays that are no regret for every player is a coarse correlated equilibrium.

It's easy to see that every Nash is a coarse correlated equilibrium. But a coarse correlated equilibrium p induces a Nash equilibrium if there exists a probability distribution p_i for every player i such that for every $\bar{s}$, $p(\bar{s})$ can be expressed as $\Pi_i p_i(\bar{s}_i)$.

We look at the example of Rock-paper-scissors to find a coarse correlated equilibrium. The following table describes the payoff where (x, y) denotes that the payoff to the row player is x and that column player is y.

	R	P	S
R	(0,0)	(-1,1)	(1,-1)
P	(1,-1)	(0,0)	(-1,1)
S	(-1,1)	(1,-1)	(0,0)

This game admits a unique Nash equilibrium, the mixed Nash of choosing one of the three strategies at random.

A uniform distribution on $(R,P),(R,S),(P,R),(P,S),(S,R),(S,P)$, that is, the non-tie strategy vectors, is a coarse correlated equilibrium. We can see that if any player chooses a fixed strategy, his expected payoff will stay the same i.e. 0.

If we change the payoff table to the following

	R	P	S
R	(-2,-2)	(-1,1)	(1,-1)
P	(1,-1)	(-2,-2)	(-1,1)
S	(-1,1)	(1,-1)	(-2,-2)

then the same is a coarse correlated equilibrium, where the expected payoff per player is 0. Here choosing a fixed strategy will decrease any player's payoff to -2/3.

This modified game too has a unique Nash which is choosing each strategy uniformly at random, giving each player a negative payoff of -2/3.

Here we note that in this example the coarse correlated equilibrium is uniform over a set of strategy vectors that form a best response cycle.

We now define a correlated equilibrium.

Definition. A correlated equilibrium is defined as a probability distribution p over strategy vectors such that for every player i, and every strategy $s_i \in S_i$

$$\Sigma_{\bar{s}} p(\bar{s}|\bar{s}_i = s_i) c_i(\bar{s}) \leq min_{x \in S_i} \Sigma_{\bar{s}} p(\bar{s}|\bar{s}_i = s_i) c_i(x, \bar{s}_{-i})$$

Intuitively, it means that in such an equilibrium, every player is better off staying in the equilibirum than choose a fixed strategy, when all the other players assume that this player stays in equilibrium. Staying in equilibrium hence can be thought of as following the advice of some coordinator. In other words, when other players assume you follow your advice, you are better off following the advice than deviating from it.

We consider as an example the game of Chicken. Two players play this game, in which each either Dares to move forward or Chickens out. If both Dare, they will crash, if one Dares then he wins and the other loses and if none Dare then no one wins. The payoffs are as follows.

	D	C
D	(-10,-10)	(1,0)
C	(0,1)	(0,0)

This game has three Nash equilibra, two are pure and one is mixed. The pure equilibra are (D,C) and (C,D). The mixed Nash is choosing to Dare(D) with probability large enough to drive down the other player's expected payoff if he chose to just Dare, and small enough to ensure that just Chickening is not a better option.

A correlated equilibrium would be the uniform distribution over $(D, C), (C, C), (C, D)$. We can think of the co-ordinator as a traffic light to each player. A player can view his light but not the other player's. If a player is told to Chicken, it's possible (with probability $1/2$) the other has been told to Dare, hence it's better to Chicken. If a player is told to Dare, the other player has been told to Chicken, and hence it's best to Dare.