Lecture 34 Scribe Notes

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1 Bandwidth Sharing

Many users want to use a limited resource, how do we allocate this resource as efficiently as possible?
We define the problem:

- user i has utility $U_i(x)$ for x amount of bandwidth
- user i pays w_i for x amount of bandwidth and recieves net utility of $U_i(x) - w_i$
- the total amount of bandwidth available is B

We will also use assumptions regarding $U_i(x)$

- $U_i(x) \geq 0$
- $U_i(x)$ is increasing and concave
- $U(x)$ is continuous and differentiable (not neccessary, but useful for convenience)

The assumptions of $U_i(x)$ being increasing and concave implies that more of x is better, but two times of x doesn't imply twice the utility.

2 How to we allocate the resource optimally?

2.1 First idea: set a price p and let everyone individually optimize their own welfare

e.g. each player individually finds
$\arg\max_x U_i(x) - px$
Solving for this maximum yields
$U_i'(x) - p = 0$
which simplifies to
$U_i'(x) = p$
This function is monotone decreasing.
Of course, if every player maximizes this function individually, it could result in players cumulatively asking for more of the resource than available. Thus, we define a market clearing price p:

2.1.1 Definition of Market Clearing

p is market clearing if there exists amounts $x_i ... x_n$ such that x_i maximizes $U_i(x) - px$ and $\sum_i x_i = B$ where B is the total amount of the resource
If $U_i(x)$ is only non-decreasing, then some users will not want any of the resource, even at price $p = 0$.
In this case, $\sum_i x_i \leq B$ by the property of free disposal; if property is not valuable, one can dispose of it for free.

2.1.2 Market Clearing Price Lemma

If a market clearing price exists, division of B into $x_i ... x_n$ is socially optimal.
$\sum_i U_i(x)$ is the max among all ways to divide B.

2.1.3 Proof of Market Clearing Price Lemma

Let $x_1^*...x_n^*$be the optimal amounts
We know $U_i(x_i) - px_i \geq U_i(x_i^*) - px_i^*$ because $U_i(x_i) - px_i$ was defined to be the maximum.
$\Rightarrow \sum_i U_i(x_i) - p\sum_i x_i \geq \sum_i U_i(x_i^*) - p\sum_i x_i^*$
We know that $p\sum_i x_i = pB$ since $\sum_i x_i = B$
$\Rightarrow \sum_i U_i(x_i) \geq \sum_i U_i(x_i^*) + p(B - \sum_i x_i^*)$
$B - \sum_i x_i^*$ is zero at optimal because all of B is allocated to users

2.1.4 Does such a p exist?

We can always find a market clearing p using the following algorithm:

- set $p = 0$ and if $\sum_i x_i(p) \leq B$, then we are done
- else, raise p until $\sum_i x_i(p) = B$

Will the price rise forever?
This will not happen because of the bounded derivative (marginal utility). If the price was ever raised higher than $U_i^{'}(\frac{B}{n})$, then users will only want to buy at most $\frac{B}{n}$ amount of bandwidth and the total amount of bandwidth requested would be less than or equal to the total amount of bandwidth.
$p = \arg\max_i U_i^{'}(\frac{B}{n})$ results in $x_i \leq \frac{B}{n}$ for all i

2.2 Another idea of optimal allocation: Fair Sharing

The game:

1. Ask every user for how much money they are willing to pay for the resource w_i
2. Collect the money
3. Distribute the resource in x_i amounts by the formula $x_i = (\frac{w_i}{\sum_j w_j}) * (B)$

This yields an effective price of $p = \frac{\sum_j w_j}{B}$

2.2.1 Is the distribution $w_1...w_n$ optimal?

Assume w_j for all $j \neq i$
Users will individually optimize. Imagine a game where every user bids their value w_i, and get some bandwidth in return. At a Nash equilibrium, each w_i is optimal given w_j for all $j \neq i$ are fixed. Thus, the following arg max is a Nash equilibrium.
$\arg\max_{w_i} U_i[(\frac{w_i}{\sum_j w_j}) * (B)] - w_i$
$U_i^{'}(\frac{w_i}{\sum_j w_j} * B)(\frac{1}{\sum_j w_j} * B - \frac{w_i}{(\sum_j w_j)^2} * B) - 1 = 0$
By the effective price and #3, we conclude
$\Rightarrow U_i^{'}(x_i)(\frac{1}{p} - \frac{1}{p} * \frac{x_i}{B}) - 1 = 0$
$\Rightarrow U_i^{'}(x_i)(1 - \frac{x_i}{B}) = p$
Comparing this to price equlibrium $U_i^{'}(x_i) = p$ we see that when $\frac{x_i}{B}$ is close to zero, the two conditions are almost the same. In the context of the internet, users usually do not have a significantly large share of the bandwidth and this pricing scheme is thus approximately optimal.