CS 6840 – Algorithmic Game Theory (4 pages)	Spring 2012

Lecture 9 Scribe Notes

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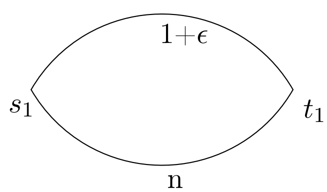
Today's topic: book Sec. 19.3. Reference: Anshelevich et al. *The Price of Stability for Network Design with Fair Cost Allocation.* FOCS 2004.

1 Network Design Games

Description:

- n players;
- each player i: connect (s_i, t_i) on a directed network $G = (V, E)$;
- strategy for player i: $P_i \in \mathcal{P}_i$;
- each $e \in E$ has a cost c_e;
- fair cost allocation: $d_e(n_e) = \frac{c_e}{n_e}$, where n_e is the number of players choosing e;
- player cost: $C_i(S) = \sum_{e \in P_i} \frac{c_e}{n_e}$;
- social cost: $SC(S) = \sum_i C_i(S) = \sum_{e \in S} n_e \frac{c_e}{n_e} = \sum_{e \in S} c_e$

Example 1: consider the following network: n players can choose either edge to connect s_1 and t_1.



One Nash is that all players choose the edge with cost $1 + \epsilon$. In this case, player's cost $C_i = \frac{1+\epsilon}{n}$. The other Nash is that everyone chooses the edge with cost n, where the player's cost $C_i = n/n = 1$.

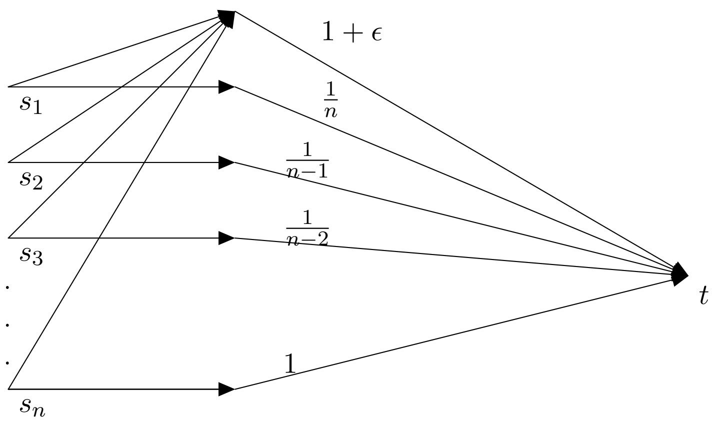
From the analysis above, PoA $\geq n$ in this class of games. On the other hand, PoA is at most n, since in an NE, a player's worse–case cost is at most $\sum_{e \in P_i^*} c_e \leq \sum_{e \in \text{OPT}} c_e$. Therefore the summation of all players' costs is upper bounded by n times of the optimal cost.

More naturally, we are interested in relation between the *best* Nash and the optimal.

1.1 Price od Stability

Definition: Price of Stability (PoS) $= \dfrac{SC(\text{Best–NE})}{SC(\text{OPT})}$

Example 2: consider the following network: Each player i wants to connect from s_i to t. The costs on edges are shown in the figure, if they have costs.



Obviously the optimal strategy has $SC(\text{OPT}) = 1 + \epsilon$, where everyone chooses the route with the $(1+\epsilon)$ edge. There is a unique Nash for this game – that is player i chooses the route with the $\frac{1}{n+1-i}$ edge. $SC(\text{U–NE}) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = H_n$. Thus, here PoS$\geq H_n = O(\log n)$. Comparing with PoA, which is n, PoS is still exponentially better.

Now we are interested in upper bounding PoS for network design games.

Notice that by definition, network design games are congestion games. Thus they are potential games with the poetential function as follows:

- $\Phi(S) = \sum_e \sum_{i=1}^{n_e} d_e(i) = \sum_e \sum_{i=1}^{n_e} \frac{c_e}{i}$.

A special Nash among all is the global minimizer of the potential. However, min potential $\neq$ min SC. In the main theorem, we present that min potential is an approximate of min SC in some sense.

Theorem 1. Let us consider a congestion game with potential function $\Phi(\cdot)$. Suppose that for any strategy S,

$$A \cdot SC(S) \leq \Phi(S) \leq B \cdot SC(S),$$

then PoS$\leq B/A$.

Proof. Let NE denote the global minimizer of the potential, which is a Nash.

$$SC(\text{NE}) \leq 1/A \cdot \Phi(\text{NE}) \leq 1/A \cdot \Phi(\text{OPT}) \leq B/A \cdot SC(\text{OPT}).$$

□

For the class of network design games, we have the following corollary.

Corollary 2. PoS of network design games is $\leq H_n$.

Proof. SC is the sum of the costs of all edges:

$$SC(S) = \sum_{e \in S} c_e.$$

The potential is by definition

$$\Phi(S) = \sum_{e \in S} \sum_{i=1}^{n_e} \frac{c_e}{i} = \sum_{c \in S} c_e H_{n_e}.$$

Therefore,

$$SC(S) \leq \Phi(S) \leq H_n \cdot SC(S).$$

Then apply Theorem 1 to prove the corollary. □

For congestion games with *linear delays*:

- $d_e(n_e) = a_e n_e + b_e$, where $a_e, b_e \geq 0$,

we have the following theorem

Theorem 3. For congestion games with linear delays as defined above, PoS ≤ 2.

Proof. The social cost is

$$SC(S) = \sum_e n_e d_e(n_e) = \sum_e a_e n_e^2 + b_e n_e.$$

The potential is

$$\Phi(S) = \sum_e \sum_{i=1}^{n} (a_e i + b_e) = \sum_e (a_e \frac{n(n+1)}{2} + b_e n_e).$$

Therefore,

$$\frac{1}{2} SC(S) \leq \Phi(S) \leq SC(S).$$

Again, apply Theorem 1 to complete the proof. □

More generally, for the class of network design games, we consider the case where the cost c_e is no longer a constant. Suppose that

- $c_e(i)$ is a concave and monotone increasing function of i, and thus that $\frac{c_e(i)}{i}$ is a decreasing function of i.

Then we have the following theorem.

Theorem 4. For the class of network design games, we assume that the building cost c_e is a concave and increasing function of n_e. Then PoS $\leq H_n$.

Proof. The social cost is

$$SC(S) = \sum_e c_e(n_e).$$

The potential is

$$\Phi(S) = \sum_e \sum_{i=1}^{n_e} \frac{c_e(i)}{i}.$$

Thus,

$$\Phi(S) \leq \sum_e \sum_{i=1}^{n_e} \frac{c_e(n_e)}{i} = \sum_e c_e(n_e) H_{n_e} \leq H_n \cdot SC(S),$$

where the first inequality follows from our assumption that $c_e(\cdot)$ is increasing. On the other hand, noticing that $\frac{c_e(i)}{i}$ is a decreasing function of i, we have that

$$SC(S) = \sum_e c_e(n_e) = \sum_e \sum_{i=1}^{n_e} \frac{c_e(n_e)}{n_e} \leq \sum_e \sum_{i=1}^{n_e} \frac{c_e(i)}{i} = \Phi(S).$$

By applying Theorem 1 we complete the proof. □

For Example 2, if we remove the directedness, the Nash would be that everyone goes through the cheapest edge, which is also the optimal. Then the H_n bound is no longer tight. In fact, when the underlying graph is undirected, it is an open question that whether there is a constant PoS instead of H_n. The best lower bound is ≈ 2.24.

Can we compute the best Nash? Unfortunately computing the best Nash is NP–hard. Computing the Nash that minimizes the potential is also NP-hard.