COM S 6840 – Algorithmic Game Theory March 28, 2012

Lecture 28: Greedy Algorithm in General Combinatorial Auctions

Instructor: Eva Tardos *Scribe: Adith Swaminathan*

1 Announcements

Extra credit is a determiner for A+ grade, or to recover from sub-optimal performance in another Homework. Remember, improving the articles on Wikipedia about Algorithmic Game Theory is worth extra credit.

2 Review: Greedy Algorithm as a Mechanism

- *Lecture 26*: In a *matroid* environment, the greedy algorithm was essentially VCG. Hence, the mechanism was optimal and telling the truth was a dominant strategy for the players.
- *Lecture 27*: For combinatorial auctions with *single-minded players*, a greedy algorithm achieves a $\sqrt{n}$-approximation to the social welfare of the optimal allocation (n being the number of players). *Critical value payments* were introduced, and these ensure that telling the truth remains a dominant strategy for the players.
- *Today*: For combinatorial auctions with no single-mindedness constraints on the players, if we have a greedy algorithm that achieves a c-approximation, then all pure Nash equilibria have social value $\geq 1/(c+1)$ of the optimal allocation.

3 Set-Up

Recall the definition of a combinatorial auction. Set S of items on sale. n bidders, each bidder i has $v_i(A) \geq 0$ for any subset $A \subseteq S$. In the previous lecture, we introduced *single-minded* valuations v_i which meant, each bidder had a desired set A_i and $v_i(A) = v_i$ if $A_i \subseteq A$ and 0 otherwise. We generalize this assumption for today: we merely assert that v_i is a monotone non-decreasing set function for each bidder i. Remember, for a mechanism that solicits bids, computes allocations and charges payments, there were three issues:

- Solicit bids: Potentially exponentially (in $|S|$) many bids to be submitted by each bidder, one for each subset $A \subseteq S$. We sidestep this issue today by asking bidders to bid for only as many sets as they choose.

- Compute allocation: Finding an optimal allocation given bidders' submitted bids remains an instance of the *set-packing* problem. We shall use a greedy algorithm, but our analysis shall be ignorant of the precise quantity that the algorithm is greedily optimizing at each step.
- Charge payments: We shall define $\theta_i(A)$ to denote the *critical value* of a set A for bidder i, as a natural extension of the notion of critical value introduced in the previous lecture. Payments shall be these *critical values* for the allocated sets.

Note that bidder i bids $b_i(A)$ for any sets as she feels like, hence b_i need not be monotone non-decreasing. Consider the following greedy algorithm. At each step, it looks at the value of a function $f(v, A)$ where v is a bid value, and A is the set for which that value was bid, and greedily allocates to the maximizer. Two conditions on f: it must be monotone increasing in v and monotone decreasing in $|A|$. It is a simple exercise to verify that the greedy algorithms shown in the previous lectures have this property for f.

Algorithm 1 for greedy social welfare maximization in a combinatorial auction.

$i \leftarrow 1$ {A_i shall be the set allocated to bidder j_i in iteration i.}
$S \leftarrow \phi$ {to keep track of all allocated items so far.}
for Each bidder j **do**
  $x[j] \leftarrow$ **false** {to keep track of which bidders have already been allocated sets.}
end for
for $i = 1 \to \infty$ **do**
  $(A^*, j^*) \leftarrow \underset{(A,j) \text{s.t.} x[j]=\textbf{false}, A \cap S = \phi}{\operatorname{argmax}} f(b_j(A), A)$
  if A^* is ϕ **then**
    return S
  end if
  $A_i \leftarrow A^*$
  $j_i \leftarrow j^*$
  $x[j_i] \leftarrow$ **true**
  $S \leftarrow S \cup A^*$
end for
return S

With this allocation algorithm in hand, we shall now examine how to set prices. If set A_i was awarded to bidder j_i in the i^{th} iteration of the algorithm, we can define $\theta_{j_i}(A_i)$ as the smallest (positive) bid by bidder j_i on which she continues to be allocated A_i. Note that this θ_{j_i} is independent of the bidder's personal valuation for the set. Moreover, it is easy to compute this θ_j efficiently. Hence, for any bidder j, we can define $\theta_j(A)$ for each $A \subseteq S$ as the smallest bid by j that would have ensured that she were allocated the set A by the greedy allocation scheme. If j was allocated A by the greedy allocation, $\theta_j(A) \leq b_j(A)$, and $\theta_j(A) \geq b_j(A)$ otherwise.

4 Price of Anarchy bound

Theorem 1 *Lucier and Borodin, 2010*
If a greedy algorithm achieves an allocation that is a c-approximation to the social optimum, and charges critical value prices, assuming no bidder in the combinatorial auction bids above their value for any set, for a pure Nash equilibrium, its social value $\geq 1/(c+1)$ *Optimal*

Proof. In the Nash equilibrium, let A_i be the set allocated to bidder i, and let O_i be the set allocated to i in the socially optimal allocation. Every bidder need not have a set allocated, denote those sets by ϕ.

$$[O_i] = \underset{[S_i]\text{s.t.}S_i\text{disjoint}}{\text{argmax}} \sum_i v_i(S_i) \tag{1}$$

This is the definition of the socially optimal allocation.
Fact: The greedy algorithm is a c-approximation. This means,

$$\sum_i b_i(O_i) \leq c \sum_i b_i(A_i)$$

Note that the bidders need not have submitted any bids for O_i at all (ie, $b_i(O_i) = 0$). We shall assert the stronger claim:

$$\sum_i \theta_i(O_i) \leq c \sum_i b_i(A_i) \tag{2}$$

Proof. of (2)
For some bidder i, if $O_i = A_i$, by the definition of θ, we are assured that $\theta_i(O_i) \leq b_i(O_i) \leq cb_i(O_i)$.
Consider running the greedy allocation algorithm on an instance of the auction where bidders who were allocated sets $A_i \neq O_i$ originally, submit modified bids b': for a bidder i, for any set $\neq O_i$, their bid is unchanged, and for O_i, they submit a modified bid of $\theta_i(O_i) - \epsilon$. Notice that by definition of θ_i, this bid is insufficient to win O_i for bidder i, the greedy algorithm is unaffected and allocates A_i to bidder i as before. Remember however that the algorithm is c-approximate. Hence,

$$c \sum_i b_i(A_i) \geq max[S_i]\,\text{s.t.}S_i\text{disjoint} \sum_i b'_i(S_i) \geq \sum_i (\theta_i(O_i) - \epsilon) \tag{3}$$

The first inequality is by definition of a c-approximate algorithm, the second inequality comes from the way b' is defined. Considering the case when ϵ becomes vanishingly small proves the claim.
Fact: All players bid below their value for any set.

$$b_i(A_i) \leq v_i(A_i) \tag{4}$$

Fact: The submitted bids by all players form a Nash equilibrium. Consider the following deviation from this equilibrium. For bidder i, she retracts all her bids and only bids for O_i with a bid of $\theta_i(O_i)+\epsilon$. Note that by definition of θ_i, she is guaranteed to be allocated O_i, and hence derives a value of $v_i(O_i)-\theta_i(O_i)$. Since it is a Nash equilibrium, the bidder must necessarily derive a smaller or equal value from this deviation than before. Hence,

$$v_i(O_i) - \theta_i(O_i) \leq v_i(A_i) - \theta_i(A_i) \leq v_i(A_i) \tag{5}$$

the last inequality follows since θ is, by definition, positive. Summing over all bidders i,

$$\begin{aligned} \sum_i v_i(O_i) \leq & \quad \sum_i (v_i(A_i) + \theta_i(O_i)) && \text{from (5)} \\ \leq & \quad \sum_i (v_i(A_i) + cb_i(A_i)) && \text{from (3)} \\ \leq & \quad (c+1) \sum_i v_i(A_i) && \text{from (4)} \end{aligned}$$

Hence proved.

5 Comments

Where did we use the fact that the algorithm is greedy? When arguing about the modified auction with bids as b'_i, we needed to assert that bidding just below the critical value left the allocation unchanged.

Is the conservative bidders assumption (4) reasonable? Bidding $b_i(A) > v_i(A)$ is dominated by $b_i(A) = v_i(A)$, so the assumption is indeed reasonable. Was it really necessary? Yes, since the second price auction for a single item is a special case of this set-up, and we have seen that there exist arbitrarily bad Nash equilibria in those auctions (eg, when one player bids a very high value, and all other players bid 0). Hence, for the price of anarchy result, this assumption was crucial.

Next lecture, we shall examine another simple mechanism for independent item auctions in this framework.