Lecture 15 Scribe Notes

Instructor: Eva Tardos *Jesseon Chang (jsc282)*

1 Lecture 15 – Friday 24 February 2012 - Single Item Auctions

1.1 Single Item Auctions

- n players
- Player i has value v_i for the item.
- If player i wins the item then the social value is v_i.

1.2 Second Price Auction

- Each player bids a value/willingness to pay b_i.
- Select i such that $\max_i b_i$ and make him/her pay $p_i = \max_{j \neq i} b_j$.

Property 1: A second price auction is truthful. For each player i, bidding $b_i = v_i$ dominates all other bids.

If a player i bids $b_i < v_i$ and $b_i < \max_{j \neq i} b_j < v_i$, i will want to deviate. If a player i bids $b_i > v_i$ and $v_i < \max_{j \neq i} b_j < b_i$, i will want to deviate.

Nash Equilibria?

- $b_i = v_i$ for all i is a Nash and maximizes social welfare.
- There exists other equilibria where player i with the maximum v_i makes a bid greater than the second largest value and smaller than v_i. $\max_{j \neq i} b_j < b_i < v_i$.
- Yes, there exist Nash equilibria that are not socially optimal. For example, for two players: $v_1 < v_2$, $b_1 > v_2$ and $b_2 = 0$.

All equilibria where $b_i \leq v_i$ for all i are socially optimal.

Proof: If winner i has $b_i < v_i$ and $\exists j : v_j > v_i$, the solution is not a Nash equilibrium, as j wants to deviate and outbid i. Thus there cannot exist a Nash equilibrium where the player with the highest value does not win.

1.3 English Auction

- Raise price of item slowly.
- Once there is only one player still interested, that player wins.
- Players are no longer interested once the price equals their value of the item.
- Similarly to second price auction, winner pays an amount equal to the second highest value.

1.4 Posted Price Auction

- Post a price p.
- Sell to anyone at price p if $b_i > p$.
- If a full information game, $p = \max_i v_i - \epsilon$.

Full information game is unrealistic. We consider a Bayesian game.
Bayesian Game:

- Players draw values v_i from a known probability distribution.
- Each v_i is independent and drawn from the distribution.
- An example: $v_i \in [0, 1]$ uniform
- In a second price auction, for any i $\Pr(i \text{ wins}) = \frac{1}{n}$.
- Set the sell price p such that $\Pr(v > p) = \frac{1}{n}$. If $v_i \in [0, 1]$ uniform, then $p = 1 - \frac{1}{n}$.

Theorem: Assuming values are drawn independently from identical distributions, this fixed price auction results in: $\text{E(value for winner)} \geq \frac{e-1}{e} \text{Ex}(\max_i v_i)$.

Expected value of our auction:

- The probability that the first player doesn't win is $1 - \frac{1}{n}$ by our choice of price.
- The probability there is no winner is $(1 - \frac{1}{n})^n$.
- The expected value for the winner is $\text{Ex}(v|v \geq p)$.
- Expected value of our auction is $(1 - (1 - \frac{1}{n})^n) \text{ Ex}(v|v \geq p) \approx (1 - \frac{1}{e}) \text{ Ex}(v|v \geq p)$.

Fact: We can bound the expected value of the auction above by the value of the optimal auction. We consider an auction where the seller has an unlimited number of items to sell and a player has a $\frac{1}{n}$ chance of winning. We call this auction the unlimited auction.

value of optimal auction $\geq$ max value in unlimited auction $= n(\frac{1}{n}) \text{ Ex}(v|v \geq p) = \text{Ex}(v|v \geq p)$. Thus the value of our auction above is bounded by $\text{Ex}(v|v \geq p)$.