Algorithmic Game Theory	**Final**
CS 6840 Spring 2010	

You have 7 days to do the problem from the day you started. But need to finish the latest by the end of day on May 12th. The final is not cooperative. You can only discuss the questions of the final with Renato, Hu Fu, or Eva. We will have office hours each week day, see web for the list. You may use any fact we proved in class, and any fact you find in the chapters of the book we covered, without proving the proof or reference. However, you may not use other published papers, or the Web to find your answer.

A full solution for each problem includes proving that your answer is correct. Please start by explaining what is the high-level idea of the solution (the main insights/nontrivial things necessaries to solve them problem). If you think its useful you may add also pseudocode for details. Do not submit a code only. It can make the solutions more readable if you introduce convenient notation and use it.

If you cannot solve a problem, write down how far you got, and why you are stuck.

Solutions can be submitted on CMS, or handed in at Eva's office. Please type your solution, to make it easier to read.

(1) The following game is a simple model of how companies may invest in research. There are n players (the n companies), for simplicity assume they make a binary decision whether or not to have a research division. For a player i there is a cost c of having a research division, and there is an expected benefit of v of innovation. However, the company may also learn from successful innovations of nearby companies, though the benefit is less (due to delays in learning about the innovation). Assume that we also have a graph G whose nodes are the players. The game is as follows. Each player has two strategies: deciding whether to invest in research. A company i has cost c if it invests in research, and 0 cost otherwise. The company has benefit v if it invested in research, benefit γv if it didn't invest in research, but one of its neighbors in G does, and no benefit, if no neighbor invested in research. The value for player is benefit minus cost, and social welfare is the sum of the values of all players. Assume that $(1-\gamma)v < c < v$.

(a) Show that this game is guaranteed to have pure Nash equilibria, and give a polynomial time algorithm to find one.

(b) What is the price of anarchy of this game? Express the price of anarchy in terms of the given parameters: number of players, etc. Make sure your bound is asymptotically tight, but do not worry about constant factors.

(c) This is an extremely simplified model of research. Add a feature to make the game a bit more realistic model. Does your change make the price of anarchy worse or better (prove your answer).

(2) Consider the one commodity special case of the nonatomic selfish routing game discussed in class where all traffic goes from a common source s to a common destination t. Assume for this problem that the delay on each edge is a nonnegative, **linear** and monotone increasing function of the load. The players are selecting paths with minimal delay. For this problem we evaluate solutions with the objective of minimizing the longest paths that carries flow, and we define a flow to be fair if all flow is carried on equal length paths. (This definition assumes that users realize the existence of a better path only by seeing other users who use that path, and the length of path not carrying flow is not relevant for the definition.) The Nash flow is fair (by definition). From the Braess paradox example, we also see that there can be a fair flow that is better than the flow at Nash equilibrium.

(a) Prove that the Nash flow is at most a factor of 4/3 worse than the optimal for the objective of minimizing the longest path carrying flow.

(b) For this part consider a flow f^* that minimizes average delay, that is, minimizes the usual social welfare objective function: $\sum_e f^*_e \ell_e(f^*_e)$. We know that this optimal flow may not be fair. We measure the unfairness of this flow by the ratio of the lengths of the longest and shortest (s,t) path that carries flow. Prove that the unfairness of the flow f^* is at most 2.

(3) In class we considered the network formation game, where players were trading off cost of edges they play for with a desire to form a connected network with small distances. Here we consider a different version where players only care to keep the network connected, but are not sensitive to distances. Let G be a graph whose nodes are the players, and edges are possible edge that can be bought. Each edge e has a cost $c_e \geq 0$. The strategy of each node v is to select one or more edges adjacent to node v, and pay for the cost of setting up these edges. The cost for each player is ∞ if the union of the bought edges do not form a connected graph, and the sum of the costs of the edges they bought, if the graph is connected. The social cost is the sum of the player's costs.

(a) show that the price of stability for this game is 1, i.e., there exists a Nash equilibrium that is optimal.

(b) Is it possible to bound the price of anarchy in terms of the graph parameters (number of edges, and nodes)?

(4) We now consider a variant of the atomic selfish routing game with k players. We have a graph G and a delay function $\ell_e(x)$ that is monotone increasing and convex for each edge $e \in E$. Player i has a source s_i and a destination t_i, and would like to select an $s_i - t_i$ path P_i on which to route 1 unit of traffic. Player i will tolerate up to d_i delay, but prefers to not be routed to having more than d_i delay.

(a) Show that this game always has a pure (deterministic) Nash equilibrium.

(b) We have evaluated such routing games is with the sum of all delays as cost. However, in this version, the cost may be low simply because few players get routed. Thus we can instead consider the value gathered by each player; d_i minus the delay incurred if i does route her traffic, and 0 if she doesn't. By definition, all players routed have nonnegative value at equilibrium. The total value of a solution is simply the sum of player values. Show that this is a utility game (of Section 19.4.3). Please note that Chapter 19.4.3 of the book has an error in defining utility games. Should require (i) only when $s \notin S'$. When $s \in S' \setminus S$ the resulting inequality is the monotonicity, that only monotone utility games need to satisfy.

(c) Is this game a monotone utility game?

(5) We considered the proportional sharing mechanism of on a single link ((of Feb 15-17). Now extend the fair-sharing mechanism to a pair of resources, say with capacity C_1 and C_2 and 3 users. Users 1 and 2 wants only resource 1 and 2 respectively, and have a utility function $U_i(x)$ for the resource, while user 3 wants the same amount of both. So if allocated amounts y_i and y_2 his/her utility is $U_3(min(y_1, y_2))$. Assume that U_i is continuously differentiable, strictly monotone increasing and concave. There are two variants of this game. Users 1 and 2 offer amount w_i of money for the resource they want, user 3 offers amounts w_3^1 and w_3^2 for the two resources separately. Each resource is then allocated using fair sharing: $x_1 = C_1 * \frac{w_1}{w_1+w_3^1}$ for user 1, and $y_1 = C_1 * \frac{w_3^1}{w_1+w_3^1}$, and similarly for resource 2.

(a) This allocation is very easy to implement, but it possibly allocates different amounts to user 3 of the two resources, which is not good for user 3. Show that this will not happen in equilibrium, that is $y_1 = y_2$ in any equilibrium.

(b) Give the conditions of equilibrium for this game, and argue that equilibrium must exists.

Even though user 3 will get the same amount of the two resources at equilibrium, one may view this game as awkward, as it can allocate different amounts on the two edge to user 3. An alternate version would be the following. Users 1 and 2 offer amount w_i of money for the resource they want, user 3 offers amounts w_3 for the two resources combined. We then solve the equation system $w_3^1 + w_3^2 = w_3$ and $C_1 * \frac{w_3^1}{w_1+w_3^1} = C_2 * \frac{w_3^2}{w_2+w_3^2}$. Then use w_3^1 and w_3^2 in the fair sharing mechanism, as above. Note that this system always allocates the same amount of both resources to user 3, but its harder to compute the allocation.

(c) Assume that $C_1 = C_2 = C$ and $U_1 = U_2$ for this part. Give the conditions of equilibrium for this new game, and argue that equilibrium must exists. (Note that without some extra assumption the equilibrium does not always exists.)

(d) This part is attempting to explore which of the two rules would the players prefer. Give an intuitive explanation of your finding. Consider the further special case when the utility of all players is $U_i(x) = x$. What are the optimal allocation, and the Nash allocations in the two different games. Which player prefers which of the rules?

(6) Assume for this problem that utilities of all players are $0 \leq u_i(s) \leq 1$ for all strategy vectors s. Recall that for a player i a strategy s_i is ϵ-dominated by another strategy s_i' if for all strategy vectors s_{-i} the utility $u_i(s_i, s_{-i}) \leq u_i(s_i', s_{-i}) - \epsilon$. We have seen that playing the weighted majority the probability of playing an ϵ-dominated strategy is going to zero, but that a coarse correlated equilibrium can have no-regret, and yet play an ϵ-dominated strategy with positive probability (PS 3 problem 5).

(a) can there be a Nash equilibrium when a player is playing an ϵ-dominated strategy with positive probability?

(b) Is the first statement also true when playing regret matching? i.e., does the probability of playing an ϵ-dominated strategy is going to zero, when playing regret matching? For answering this question assume that playing regret matching we update the history with the expectation (i.e., for simplicity assume that expected outcomes always happen). Recall that regret matching works as follows. We keep a history of plays (sequence of strategy vectors selected in each step. This defines an empirical probability distribution of strategy vectors (the frequency a vector was selected). Each player computes his/her regrets for this distribution, and independently selects a strategy for the next step that is proportional to the current value of regret. Then we add the strategy vector selected to the history. For this problem assume you add the expected distribution, i.e., for each strategy vector s you add the probability of being selected (rather than randomly selecting one). Recall that we proved that (this version of) regret matching guarantees that the play converges to the set of coarse correlated equilibria, i.e., for any $\delta > 0$ after long enough time the empirical probability distribution is within δ distance of the set of coarse correlated equilibrium set.)