

(1) We have considered the load balancing problem with uniform machines (as described in Chapter 20.4). We use  $w_j$  for the weight of job  $j$ , and  $L_j$  the random variable of load experienced by job  $j$ . We considered the egalitarian objective function (or makespan,  $OPT = \min \max_j L_j$  where the  $\min$  is taken over all possible assignments of jobs to machines. We showed that  $\max_j E(L_j) \leq 2OPT$  and  $E(\max_j L_j) \leq O(\log m)OPT$ . In this problem we want to consider the social welfare objective  $\sum_j L_j$ . (Note that by linearity of expectation  $E(\sum_j L_j) = \sum_j E(L_j)$ .) Give a constant bound on the price of anarchy for this game considering the social welfare objective.

(2) It is a major outstanding open problem whether a Nash equilibrium of a game can be found in polynomial time. In this problem, we consider the special case with two players. To be more formal, assume that there are 2 players, and player  $i$  chooses between  $n_i$  pure strategies. Assume that the game is given by listing the payoff for each player for each  $n_1 \times n_2$  possible plays (this is the traditional payoff matrix). Give a finite algorithm for finding a (**possibly mixed**) Nash equilibrium for this case. Your algorithm may run in exponential time.

(3) Consider a 2-person game in the matrix form also used in the previous problem. Assume that both players have  $n$  pure strategies. In a Nash equilibrium a player may be required to play a mixed strategy that gives non-zero probability to all (or almost all) of his pure strategies. Strategies that mix between so many pure options are hard to play, and also hard to understand. The goal of this problem is to show that one can reach an almost perfect Nash equilibrium by playing strategies that only choose between a few of the options.

As in class, we will use  $p^j$  to be the probability distribution for player  $j$ , so  $p_i^j$  is the probability that player  $j$  will use his  $i$ th pure strategy. The support of a mixed strategy  $p^j$  for player  $j$  is the set  $i$  such that  $p_i^j > 0$ , i.e., the number of different pure strategies being mixed. We will be interested in solutions where each job has a strategy with small support. Unfortunately, some Nash equilibria may give players strategies with large support.

For a given  $\epsilon > 0$ , we will say that a set of mixed strategies  $p^1, p^2$  is  $\epsilon$ -approximate Nash if for both players  $j = 1$  or  $2$ , and all other strategies  $\bar{p}^j$  for this player, the expected payoff for player  $j$  using strategy  $\bar{p}^j$  is at most  $\epsilon M$  more than his expected payoff using strategy  $p^j$ , where  $M$  is the maximum payoff.

Show that for any fixed  $\epsilon > 0$  and any 2 player game with all nonnegative payoffs, there is an  $\epsilon$ -approximate Nash equilibrium such that both players play the following simple kind of mixed strategy: For each player  $j$  the strategy selects a subset  $\hat{S}_j$  of at most  $O(\log n)$  of player  $j$ th pure strategies, and makes player  $j$  select one of the strategies in  $\hat{S}_j$  uniformly at random. The set  $\hat{S}_j$  may be a multi-set, i.e., may contain the same pure strategy more than once (such a strategy is more likely to be selected by the random choice). The constant in the  $O(\cdot)$  notation may depend on the given parameter  $\epsilon$ .

Hint: consider any mixed Nash strategy with possibly large support, and try to simplify the support by selecting the subsets  $\hat{S}_j$  for the two players based on this Nash strategy.

(4) In the Vetta facility location game we showed that if the facilities cost 0, and each player is allowed to locate one facility, than for any Nash equilibrium we get that  $value(OPT) \leq 2 \cdot$

$value(Nash)$ , where  $OPT$  denotes the solution of maximum value.

In this problem, we consider a variant of this game where facilities cost money. In this variant, each possibly facility  $j$  has a cost  $c_j$ , and locating a facility at  $j$  the player has to pay cost  $c_j$ . (We define prices  $p_i$  as before, the value to a user who gets served at price  $p_i$  at facility  $j$  is  $\pi_i - c_{ij} - p_i$  as before, and in defining the value derived by a player who located at facility  $j$ , we use  $(\sum p_i) - c_j$ , where the sum is over the locations that will be served by facility  $j$ , so we simply subtract  $c_j$  from the value used in the game in class).

- (a) Is the inequality  $value(OPT) \leq 2 \cdot value(Nash)$  also true for the variant of this game that facilities cost money? Prove or give an example where it is not true.
- (b) Let  $fac(OPT)$  denote the total facility cost of optimum, that is the sum  $\sum c_j$  over the facilities  $j$  opened by the optimal solution. This can be viewed as the investment cost needed for the optimal solution. Show that in any Nash equilibrium of this game satisfies  $value(OPT) \leq 2 \cdot value(Nash) + fac(Nash)$ . One can read this inequality to say that the price of anarchy can be bad (much worse than a factor of 2) for equilibria that need high investment cost.
- Is the inequality  $value(OPT) \leq 2 \cdot value(Nash) + fac(Opt)$  also true?

(5) Consider an  $n$  person game where each player has only two strategies. This game has  $2^n$  possible outcomes (for the  $2^n$  ways the  $n$  people can play, so even giving the game in the above matrix form is rather problematic (takes exponential information)). One special class of games that one can consider is called graphical games. The idea is that the value (or payoff) of a player can only depend on a subset of players. We can define a dependence graph  $G$  whose nodes are the players, and we put an edge between two players  $i$  and  $j$  if the payoff of player  $i$  depends on the strategy of player  $j$  or vice versa. This way if node  $i$  has only some  $k$  neighbors, than his payoff only depends on his own strategy and the strategy of his neighbors.

For this problem assume that the dependence graph is a path and each player has two pure strategies, i.e., the payoff of each node on the path depends only on the strategies played by its (at most two) neighbors. Now the payoff values can be given by  $2^3 = 8$  entries for each node. Give a polynomial time algorithm to decide if the game has a pure Nash equilibrium. (Note that there are  $2^n$  possible pure strategies, but your algorithm must run in time polynomial in  $n$ .)

(6) Consider an  $n$  person game where each player has 2 strategies. For this problem, think of the strategies as “on” and “off”, i.e., the strategy is either to participate, or not to participate. Further assume that all players have the same payoff functions, and that the payoff for a player only depends on whether this player is “on”, and the number of people playing strategy “on”. So the game is defined by  $2n$  values:  $u_{on}(k)$  and  $u_{off}(k)$  that denote the payoff for playing the “on” and “off” strategy assuming  $k$  players chose to play “on” with  $k = 0, \dots, n$  (where  $u_{on}(0)$  and  $u_{off}(n)$  is meaningless). Give a polynomial time algorithm to find a correlated equilibrium for such a game. Note that the input to this problem consists of the  $2n$  numbers above, so polynomial means, polynomial in this input length. You may use that linear programming is solvable in polynomial time.