CS 684 Algorithmic Game Theory	**Scribe: Ian Kash**
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A Review of Non-atomic routing

We have a graph $G = (V, E)$. With each edge e, we associate a delay function $l_e(x)$ (which we assume is continuous and monotone). There are k types of users where each type of user wants to get from a source s_i to a sink t_i and there is demand $dem(i)$. A solution is a flow f such that for each path P the flow along that path satisfies $f_P \geq 0$ and for each type i, $\sum_{P.\ s_i \to t_i \text{ path}} f_P = dem(i)$. We can compute the flow on a single edge e as $f(e) = \sum_{P.\ e \in P} f_P$. The delay experienced along a path P is the sum of the delays on the edges or $l_P(f) = \sum_{e \in P} l_e(f(e))$. f is Nash if $\forall i \forall P, Q\ s_i \to t_i$ paths, $f_P > 0 \Rightarrow l_P(f) \leq l_Q(f)$.

The main goal

As discussed in previous lectures there are a number of different measures we could use to determine the quality of a flow. For today will focus on the quality being the sum of the delays ($\sum_P f_P l_P(f)$). For this measure our main goal is the following theorem:

Theorem 1 *If f is a Nash flow satisfying $dem(i)$ for $i \in 1..k$ then the quality ($\sum_P f_P l_P(f)$) $\leq$ total delay over any flow satisfying $2dem(i)$ for $i \in 1..k$*

We know that the Nash isn't optimal because of Braess' Paradox, but this result is something of a cheat. We don't say anything about how close we can get to the optimal. However we can look at this as saying that if a network is "designed well" for $2dem(i)$ then the Nash flow for $dem(i)$ will do ok.

The proof makes makes use of a new set of delay functions. Suppose we have a Nash f for the delays $l_e(x)$. Define the new functions as:

$$\hat{l}_e(x) = \begin{cases} l_e(f(e)) & x \leq f(e) \\ l_e(x) & x \geq f(e) \end{cases}$$

This new delay function is somewhat mysterious, but it will turn out to be extremely useful. A sense of what it looks like is conveyed by figure 1. In order to prove our theorem we will use four lemmas.

The four lemmas

Lemma 2 *The minimum delay $s_i \to t_i$ path with delays $\hat{l}$ in a network with no flow has the same delay as the $s_i \to t_i$ delay in our Nash (f)*

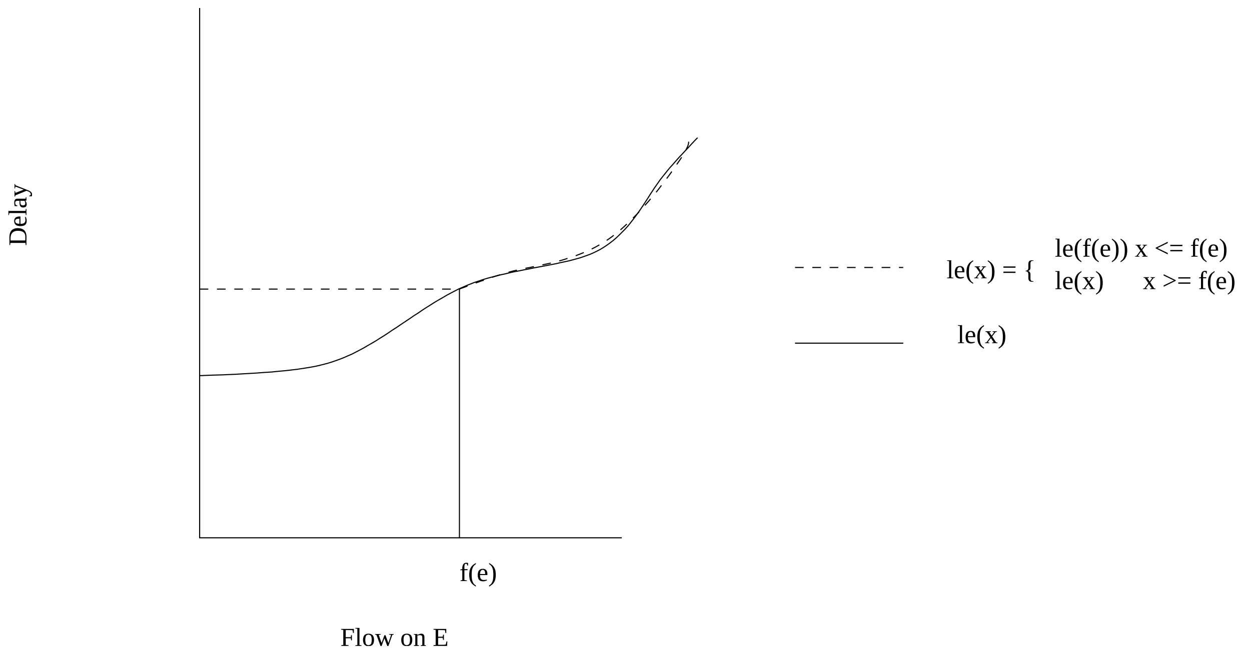

Figure 1: $\hat{l}_e(x)$

Proof. Nash flow experiences delay $l_e(f(e))$ on edge e and by the Nash property it uses shortest paths only. Since we have defined $\hat{l}$ such that $\hat{l}_e(0) = l_e(f(e))$ the delays must be the same. ■

There are two observations we can make regarding this lemma. The first is that the property that $\hat{l}_e(0) = l_e(f(e))$ is really all we care about $\hat{l}$. The definition we chose is just a natural way to do this. The second is that in a Nash the delay on all $s_i \rightarrow t_i$ paths with non-zero flow is the same and we denote this quantity L_i.

Lemma 3 *The total delay in a Nash is* $\sum_i L_i dem(i)$

Proof. This relies on the simple observation that for any $s_i \rightarrow t_i$ path either $f_P = 0$ or $l_p(f) = L_i$.

$$\begin{aligned}\sum_P f_P l_P(f) &= \sum_i \sum_{P.\ s_i \rightarrow t_i \text{ path}} f_P l_P(f) \\ &= \sum_i \sum_{P.\ s_i \rightarrow t_i \text{ path}} f_p L_i \\ &= \sum_i L_i \sum_{P.\ s_i \rightarrow t_i \text{ path}} f_p \\ &= \sum_i L_i dem(i)\end{aligned}$$

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Lemma 4 *Any flow f^* satisfying $2dem(i)$ for all i and subject to delay $\hat{l}$ has total delay at least $2\sum_i L_i dem(i)$*

From Lemma 2 we know that for an $s_i \to t_i$ path P, $\hat{l}_P(0) \geq L_i$. Since $\hat{l}_P(x)$ is monotone this also holds for any flow. Therefore

$$\begin{aligned}
\sum_i \sum_{P.\ s_i \to t_i \text{ path}} f_P^* \hat{l}_P(f^*) &\geq \sum_i \sum_{P.\ s_i \to t_i \text{ path}} f_P^* L_i \\
&\geq \sum_i L_i \sum_{P.\ s_i \to t_i \text{ path}} f_p^* \\
&\geq \sum_i L_i 2dem(i)
\end{aligned}$$

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Lemma 5 *For all flows f^* satisfying $2dem(i)$ for all i, $\sum_P f_P^* \hat{l}_P(f^*) - \sum_P f_P^* l_P(f^*) \leq \sum_i L_i dem(i)$ (the total delay in the Nash)*

Proof. Recall from last lecture that for any flow f and delays l the total delay subject to l can also be written as a sum over edges:

$$\sum_P f_P l_P(f) = \sum_e f(e) l_e(f).$$

We will use this equation for flow f^* and delays both l and $\hat{l}$.

Consider $\hat{l}_e(f^*(e)) - l_e(f^*(e))$, the difference in the delays on a single edge e. If $f^*(e) \geq f(e)$ then this is just 0 because the functions are the same. Otherwise the difference must be at most $l_e(f(e))$ because $\hat{l}_e(x)$ always takes this value in this range and $l_e(x)$ is at least 0 (see figure 2 for a picture of this case). Therefore

$$\begin{aligned}
\sum_P f_P^* \hat{l}_P(f^*) - \sum_p f_P^* l_P(f^*) &= \sum_e f^*(e) \hat{l}_e(f^*(e)) - \sum_e f^*(e) l_e(f^*(e)) \\
&= \sum_e f^*(e)[\hat{l}_e(f^*(e)) - l_e(f^*(e))] \\
&\leq \sum_e f(e) l_e(f(e))
\end{aligned}$$

This last is just another way of writing $\sum_i L_i dem(i)$

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Putting all together

All that is left to do is put the lemmas together to finish the proof. From Lemma 4 we know that the delay in f^* with $\hat{l}$ is at least $2\sum_i L_i dem(i)$. From Lemma 5 we know that the difference of the delay of f^* with $\hat{l}$ and the delay of f^* with l is at most $\sum_i L_i dem(i)$. Combining these gives that the total delay of f^* with l is at least $\sum_i L_i dem(i)$, which we know from Lemma 3 is the total delay in f.

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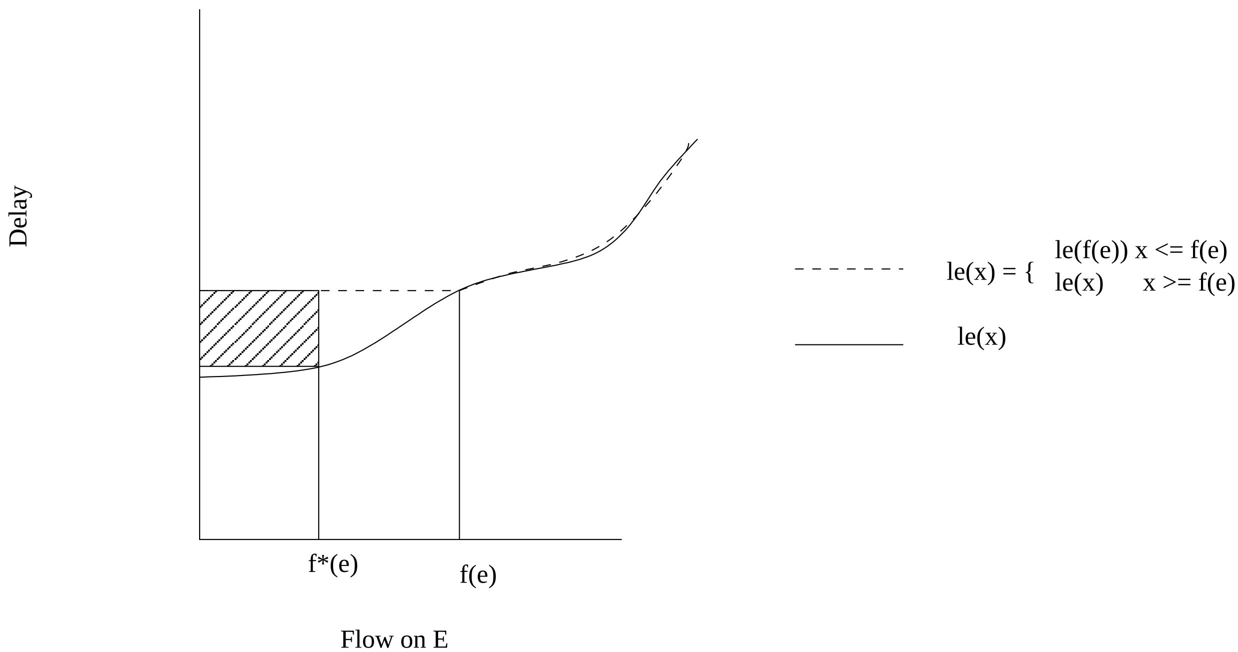

Figure 2: A graphical view of the difference