CS 684 Algorithmic Game Theory	Scribe: Gilles Noel
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This section deals with Selfish flows on networks.

Reminder : The Braess paradox Recall the Braess paradox from the first lecture. We have 1 unit of non-atomic flow that wants to go from S to T in the four node graph on the figure. The latency or delay function on edges depends on the flow as shown on the figure, constants 1 and 0 on some edges and linear $\ell(x) = x$ on the other two. Without the blue edge (U, V) the Nash equilibrium is the solution where half of the flow choose the path through U, the other choose V (red solution on the figure). The delay of this equilibrium is 1.5. But if we add the blue edge (U, V) without any delay, the new Nash equilibrium (blue solution on the figure) has all the flow going through the S, U, V, T paths, and has delay of 2.

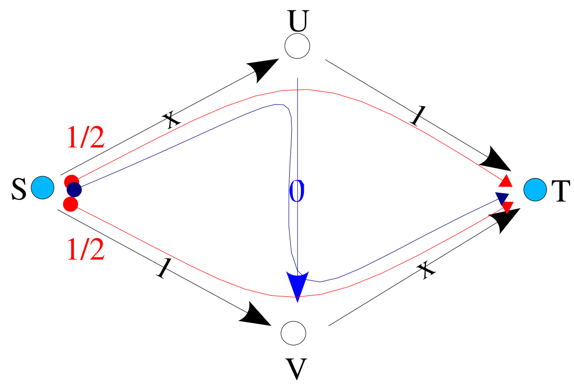

Figure 1: Braess paradox: adding edge (U, V) increases the $S - T$ delay.

Definition of selfish flows on networks The problem is defined by a directed graph $G = (V, E)$, and k types of users. User i has source s_i destination t_i and there are $dem(i)$ amount of such users. That is, users of type i want to flow from s_i to t_i in V. Each user will be infinitesimally small, with a total volume of type i users $dem(i)$.

Each edge e has a latency $\ell_e(x)$ which is a function of the flow x on e (amount of users). Assume the $\ell_e(x)$ is nonnegative, continuous and monotone-increasing function of x.

Note: We will assume $i \neq j \Rightarrow$ either $s_i \neq s_j$ or $t_i \neq t_j$. This assumption is for notational convenience only, and is no loss of generality.

Solution A solution is defined by the flow amount f_P of users following the path P. We can associate an amount f_P for all paths, but the only paths that actually will carry flow are those that connect a source s_i to a corresponding sink t_i. There may be exponentially many such paths, so it is important to describe a flow by only listing the paths that carry flow, and not the path that have $f_P = 0$. Valid flows must satisfy the following constraints.

$$\forall P, \qquad f_p \geq 0 \text{ and}$$
$$\forall i, \quad \textstyle\sum_{\{P \text{ is an } s_i \rightarrow t_i \text{ path}\}} f_p = dem(i)$$

The total flow on the edge e is given by :

$$f(e) = \sum_{P:e \in P} f_p.$$

The delay on a edge e is $\ell_e(f(e)$ effected by all the flow on the edge, and delay on a path P subject to the flow f is :

$$\ell_P(f) = \sum_{e \in P} \ell_e(f(e)).$$

Nash equilibrium We define a flow to be at Nash equilibrium if the following is true:

$\forall$ types i, all path P from $s_i \rightarrow t_i$ with $f_p > 0$ satisfies ($\forall$ path Q from $s_i \rightarrow t_i$, $\ell_P(f) \leq \ell_Q(f)$)

Corollary 1 *If P and Q are paths that carry a flow of some type i from $s_i \rightarrow t_i$ with $f_p > 0$ and $f_q > 0$, then $\ell_P(f) = \ell_Q(f)$.*

The existence of a Nash equilibrium can be proved with the potential function. We will talk discuss this in more detail next week.

Theorem 2 *A solution is a Nash equilibrium if and only if the flows minimize the potential function:*

$$\phi = \sum_{e \in E} \int_0^{f(e)} \ell_e(\xi)\ d\xi.$$

We will prove this theorem next week. For now notice that the potential function is the natural extension of the function Φ we used last class on load balancing problems. Further, by the same argument as we used there the Φ function is convex (as it is the sum of single variable convex functions).

We will consider two ways to evaluate the quality of a solution.

1. the maximum delay $\max_{P:f_p>0}(\ell_p(f))$
2. the sum of all delays $\sum_i \sum_{\{P \text{ is an } s_i \rightarrow t_i \text{ path}\}} f_P \cdot \ell_p(f)$. Note that the sum of delays is equivalent to the average delay: to get the average delay we divide the sum by the total demand $\sum_i dem(i)$. But the total demand is a constant, independent of the flow f, so the two functions are equivalent.

Finally, we note the following equivalent way to express the second objective function:

Lemma 3

$$\sum_i \sum_{\{P \text{ is an } s_i \rightarrow t_i \text{ path}\}} f_P \cdot \ell_p(f) = \sum_{e \in E} f(e) \cdot \ell_e(f(e))$$

Proof. We will simply write out the definition of $\ell_P(f)$ as a sum over edges, and the switch the order of summation.

$$\begin{aligned}\sum_i \sum_{\{P \text{ is an } s_i \to t_i \text{ path}\}} f_P \cdot \ell_P(f) &= \sum_P f_P \sum_{e \in P} \ell_e(f(e)) \\ &= \sum_{e \in E} \sum_{P: e \in P} \ell_e(f(e)) \cdot f_P \\ &= \sum_{e \in E} \ell_e(f(e)) \sum_{P: e \in P} f_P \\ &= \sum_{e \in E} \ell_e(f(e)) \cdot f(e).\end{aligned}$$

Instead of following this long line of equations, here is another way to see that the theorem holds. Consider an edge e. There is $f(e)$ amount of flow that uses this edge, belonging to a set of different path. For all flow thought this edge e, the edge contributes an $\ell_e(f(e)$ amount to the delay of the flow, no matter which path the flow is following. The delay of flows comes from delays contributed by different edges, so summing over all edges we get the sum of all delays.