CS 684 Algorithmic Game Theory	**Scribe: Yogi Sharma** (ys246@)
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1 Recap of last class

Notation Throughout this class, by the cost of a solution, we mean the total delay incurred by that solution. By cost of a flow, we mean,

$$\sum_{P \in \mathcal{P}} f_P \cdot \ell_P(f) = \sum_{e \in E} f(e) \cdot \ell_e(f(e)).$$

In the last class, we proved a theorem lower bounding the quality of Nash equilibrium in non-atomic games. The common way to bound the quality of Nash equilibrium is through the concept of "Price of Anarchy" (PoA). The price of anarchy is defined to be the worst case ratio between cost of a Nash equilibrium and cost of an optimum solution. That is

$$\text{Price of Anarchy} = \max_{N = \text{Nash Equilibrium}} \frac{\text{Cost of } N}{\text{OPT}} \tag{1}$$

We now state the theorem we proved in the last class.

Theorem 1 ([Rou 03]) *Let $\mathcal{L}$ be a class of delay functions containing all constant functions. For any $\ell \in \mathcal{L}$, let $\ell(\cdot)$ be non-negative, continuous, monotone, differentiable function and $x \cdot \ell(x)$ be convex. Furthermore, assume that $\mathcal{L}$ contains all the constant functions.*

Then, the worst case $\frac{\texttt{Nash}}{\texttt{Opt}}$ ratio (with respect to total delay in the network) for any network, any set of s_i-t_i pairs, any demand between the source-sink pairs $\texttt{dem}(i)$ is attained on a simple network having two nodes s and t, two link from s to t and demand being equal to r. One of the link carries the latency function $\ell(x) \in \mathcal{L}$ and another has a constant latency equal to $\ell(r)$.

The worst case ratio is obtained by type of networks shown in figure 1.

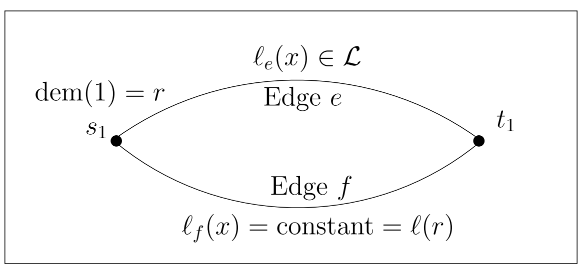

Figure 1: The worst possible ratio between the cost of a Nash equilibrium and the cost of an optimum solution is obtained by type of simple networks shown above with two nodes, two links and one commodity.

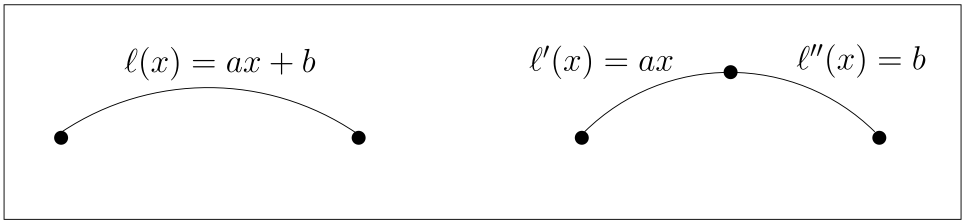

Figure 2: A general linear edge delay function can be simulated by the functions from class $\mathcal{L}$.

2 Some applications of the theorem

Theorem 1 gives us a general result about how bad the quality of a Nash equilibrium can be compared to an optimum solution. In this section, we will look at some concrete examples and calculate the worst case ratios for them. In Section 2.1, we compute the price of anarchy for networks with linear delay functions. In Section 2.2, we consider the case of (general) polynomial delay functions.

2.1 Price of Anarchy for linear delay functions

Consider a scenario when delay functions on edges of network are linear in the congestion of the edge. That means the delay function for an edge e can be written as $\ell_e(x) = ax + b$ for real numbers $a \geq 0$ and $b \geq 0$. We need the linear coefficient non-negative to ensure the monotonicity of the function (the function should be monotonically increasing), and need b non-negative to ensure that the function itself is non-negative. With these edge delays on the edges of an arbitrary network, we want to compute the worst case ratio between average delay of (worst) Nash equilibrium and average delay of an optimum solution.

Instead of considering all the possible networks, we can apply Theorem 1 and consider very simple networks with two nodes and two parallel links as shown in Figure 1. At this point, it is instructive to check that the assumptions of Theorem 1 are satisfied by linear functions with positive coefficients. So, we will consider only the simplest networks to calculate the price of anarchy.

Switch to simpler class of functions As our next step, we will replace the class of all linear functions by class of very special linear function $\mathcal{L} = \{f(x) = ax | a \geq 0\} \bigcup \{g(x) = b | b \geq 0\}$. We note that this switching is without loss of generality. Any linear edge delay function can be represented in terms of functions from class $\mathcal{L}$ by introducing an extra vertex as shown in Figure 2.

Calculation of Price of Anarchy As suggested by Theorem 1, there are two cases to consider. In one case, the latency function $\ell_e(x)$ (on the upper edge in Figure 1) is a constant function, in another, this function is a linear function.

$\ell_e(x) = b'$ Latency on two edges is equal in this case. Therefore, all the solutions are of equal cost, whether it is a Nash equilibrium or an optimum solution. So, in this case, the worst case ratio is 1.

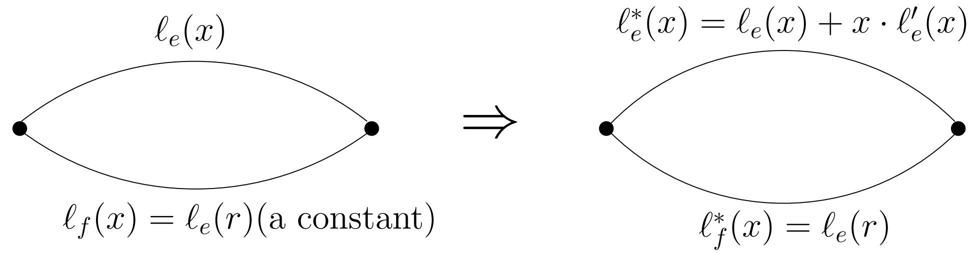

Figure 3: The problem of finding an optimal flow in a network can be reduced to finding a Nash with modified delay function. The modification needed is shown in the network on the right.

$\ell_e(x) = ax$ Let the demand be r. The delay on the lower edge (see Figure 1) is ar. A Nash solution will send all flow on the upper edge, incurring total delay of $r \cdot ar = ar^2$.

For determining the optimum solution, we note a theorem from a previous lecture which states the following.

Theorem 2 *Let G be a network with latency functions $\ell_e(x)$ on its edges. A flow in G is optimum (minimum total delay) if and only if it is a Nash flow with respect to $\ell^*_e(x) = \ell_e(x) + x \cdot \ell_e(x)$ delay functions on the edges.*

With these modified delay function, the network becomes as in Figure 3.

Nash flow in the modified network splits the flow in such a way that (modified) delay on both the links is equal. Delay on upper link is $ax + x \cdot a = 2ax$ and on the lower link is ar. In Nash flow, let the upper edge carries flow amount ζ. We have,

$$2a\zeta = ar \quad \Rightarrow \quad \zeta = r/2. \tag{2}$$

Delay incurred by the optimum solution is $\zeta \cdot a\zeta + (1-\zeta) \cdot ar = \frac{3}{4}ar^2$, while the delay incurred by Nash flow is ar^2. (All the flow goes via upper edge.) Therefore, the worst possible ratio of cost of Nash and optimal cost if

$$\frac{ar^2}{\frac{3}{4}ar^2} = \frac{4}{3}.$$

We have proved the following theorem.

Theorem 3 ([RT 02]) *The worst case ratio between the cost of a Nash solution and the optimal cost in networks with linear edge delays function is less than or equal to 4/3.*

We note that the above worst case ratio is actually achieved by simple two node, two link example. Consider the network with upper edge delay function $\ell_e(x) = x$ and lower edge delay function $\ell_f(x) = 1$, and demand of one unit. In this network, Nash incurs a cost of 1 while optimum solution splits the flow evenly between the two edges and incurs a cost of 3/4. So, the ratio is 4/3.

2.2 Price of Anarchy for general polynomials

In this section, we consider networks with edge delays being arbitrary polynomials and bound the Price of Anarchy for them. A natural question to ask is the following: if the edge delays are polynomials of degree at most p, what can we say about the Price of Anarchy? The following theorem, whose (hand-wavy) proof appear later in this section answers this question.

Theorem 4 ([Rou 03]) *The worst case ratio between cost of a Nash equilibrium and cost of an optimal solution when all the edge delay functions are polynomials of degree at most p (with all coefficients positive) is* $O\left(\frac{p}{\log p}\right)$.

We will show that it is possible to get the ratio between Nash and optimal equal to $O\left(\frac{p}{\log p}\right)$. To see this, we consider simple network with one source sink pair, two nodes, two links, and capacity equal to 1. These simplifications are without loss of generality, two node network simplification from Theorem 1 and capacity 1 simplification by a scaling argument.

Consider again the Figure 3. To consider only simple terms in the calculations, we let the upper edge have latency $\ell_e(x) = x^p$ and lower one have latency 1. The demand is 1 from s (left node) to t (right node). This simplification is again without loss of generality because a general polynomial can be split in to individual terms by a construction similar to one in Figure 2. In this case, Nash equilibrium clearly send one unit of flow on the upper edge, incurring a cost of 1. On the other hand, optimal flow splits the flow between two edges so as to minimize the total delay. By a theorem proved in previous lectures, we know that a flow is optimal if and only if it is a Nash flow with modified delay function $\ell^*(x) = \ell(x) + x \cdot \ell'(x)$ as shown in Figure 3. For this example, we have

$$\ell_e^*(x) = (1+p)x^p, \quad (3)$$
$$\ell_f^*(x) = 1. \quad (4)$$

A Nash flow with these modified delay function splits the flow so as to have equal delay on both edges. Let ζ be the amount of flow sent on the upper edge. We have $(1+p)\zeta^p = 1$ from Nash property. This solves to $\zeta = \left(\frac{1}{1+p}\right)^{1/p}$ resulting in a total delay of

$$\begin{aligned}
\text{Total delay} &= \zeta \cdot \zeta^p + (1-\zeta) \cdot 1 && (5)\\
&\approx (1-\zeta) \qquad (\text{Small } \zeta) && (6)\\
&\approx e^{-\zeta} \qquad (\text{Taylor series expansion of } e^{-\zeta} \text{ with small } \zeta) && (7)\\
&= \frac{\log(1+p)}{p}. && (8)
\end{aligned}$$

Therefore, the Price of Anarchy is $O\left(\frac{p}{\log p}\right)$.

2.3 Price of Anarchy can be arbitrarily bad

As Theorem 4 suggests, Price of Anarchy can be arbitrarily bad if we allow higher degree polynomial latency on edges. We will see an example of this.

Consider a network with two vertices, two links and demand of 1 from source to sink (see Figure 1). Let the delay on upper edge be x^p for very large p, and delay on lower edge be 1, a

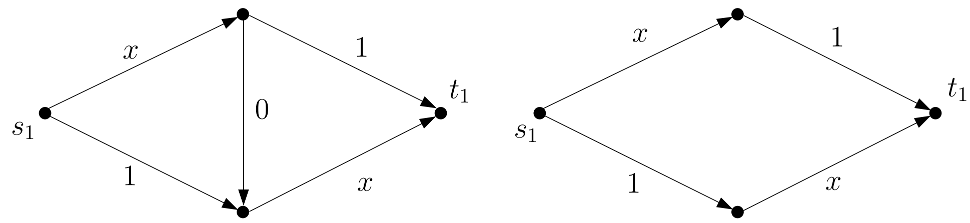

Figure 4: This figure shows the classical Braess paradox. There is a demand of 1 unit from left node to the right node. A Nash flow in the original network incurs a total cost of 2 while the cost of a Nash flow in the network after deleting the middle edge is 3/2. This decrease in cost by deleting some edges is termed as Braess paradox.

constant. Nash equilibrium will send all the demand through the upper edge, resulting in total cost of 1. On the other hand, optimum flow can send $\varepsilon \ll 1$ flow on the lower edge and $1-\varepsilon$ flow on the upper edge. The total delay of optimum solution is

$$\varepsilon \cdot 1 + (1-\varepsilon) \cdot (1-\varepsilon)^p \sim \varepsilon.$$

because $(1-\varepsilon)^p \to 0$, p being large. Therefore, this example suggests that worst case ratio can indeed be arbitrarily bad.

3 Relation to Braess paradox

We now switch topics and come back to the reason why we started the study of non-atomic games in the first place—Braess paradox. What do all these results say about severity of Braess paradox, if anything. We will see that these results indeed put an upper bound on the severity of Braess paradox. In next Subsection, we consider Braess paradox in networks with just one commodity (one source sink pair) and in Subsection 3.2, we consider multi commodity Braess paradox.

3.1 Braess paradox in single source sink pair networks

Let us consider the Braess paradox, see Figure 4 for details.

Nash equilibrium without the middle edge has cost 3/2. (It splits the flow evenly between upper and lower paths.) Counter-intuitively, Nash flow on network *with* middle edge has cost 2. (All the flow goes via upper left, middle, and bottom right edges.) The question is: can we make the ratio worse than 4/3? We will seek an answer to this question in rest of this section.

The worst case ratio in Braess paradox with linear delay functions cannot be worse than 4/3, as claimed in the following theorem.

Theorem 5 *Let G be an arbitrary network and $\hat{G}$ be obtained by adding edges to G. The worst possible ratio between cost of Nash in $\hat{G}$ and cost of Nash in G is less than or equal to 4/3.*

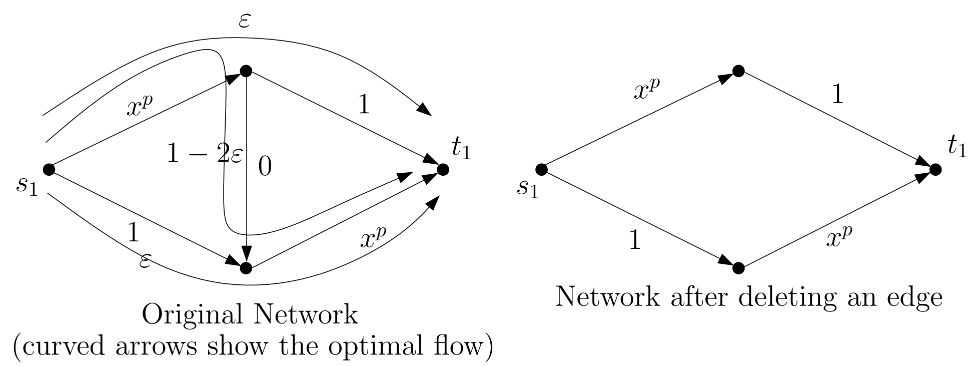

Figure 5: This figure shows the Braess paradox with polynomial edge delays. The left network is the original network with all the edges and the one on the right is obtained by deleting the middle edge. The curved arrows on the left arrow show the optimal solution for large value of p and suitably small value of ε.

Proof. We want to bound the ratio $\frac{\texttt{Nash}(\hat{G})}{\texttt{Nash}(G)}$. The bound follows from the following chain of inequalities,

$$\frac{\texttt{Nash}(\hat{G})}{\texttt{Nash}(G)} \leq \frac{\texttt{Nash}(\hat{G})}{\texttt{Opt}(G)} \leq \frac{\texttt{Nash}(\hat{G})}{\texttt{Opt}(\hat{G})} \leq \frac{4}{3}.$$

The first inequality follows as cost of an optimal solution is always less than or equal to the cost of a Nash equilibrium. Second inequality follows as the optimum values can only decrease by adding some edges. The last inequality follows from Theorem 3. ■

Braess paradox in two link networks We remark in passing (without proof) that Braess paradox is not possible in two node networks with just two links. Hence in this case, removing links cannot make the delay better (in Nash equilibrium).

3.1.1 Braess paradox in networks with polynomial edge delays

In networks with polynomial edge delays, all the bounds we proved about quality of Nash equilibrium carry over to bounds on Braess paradox. This is easily seen from the chain of inequalities in the proof of Theorem 5. (In particular, the last inequality corresponds to the bounds in Theorem 5.) But are these bounds tight? If we look at small networks, as in original Braess paradox, then the answer is no! Consider for example the network in Figure 5 with large value of p.

In the original network with added edge of capacity zero (called $\hat{G}$), Nash equilibrium has a total cost of 2 (all the flow follow the same path). On the other hand, network without the middle edge (called G) had Nash equilibrium cost slightly larger than 1. Therefore, the ratio between Nash in $\hat{G}$ and Nash equilibrium in G can be at most 2 in this simple network. But it is interesting to note that ratio worse than 2 is possible in large networks. See [Rou 01] for more details.

Also note that an optimum solution in the original network (with the middle edge) in Figure 5 does achieves a cost arbitrarily small. This can be seen by the following flow: $1-2\varepsilon$ amount goes on the zig-zag path (top left, middle, bottom right), ε amount goes on the top path, and ε goes on the bottom path. By virtue of p being large, this flow has cost close to ε.

3.2 Stronger bounds on Braess paradox

In this section, we present some bounds on the severity of Braess paradox. We will consider the case of single source sink pair and multiple source sink pairs separately. We will just see the statement of the theorems today, without proving them. We might come back to proofs later in the course if there is enough *social* pressure for doing so.

3.2.1 Severity of Braess paradox with single source sink pair

The result, in essence, states the following:

Theorem 6 ([LRT 04]) *Let G be a network and $\hat{G}$ be the network obtained by adding k links to G. This addition of edges can cause Braess paradox factor of at most $k+1$. More precisely,*

$$\frac{\texttt{Nash}(\hat{G})}{\texttt{Nash}(G)} \leq k+1.$$

In the example we considered in Figure 5), we removed one edge and worst factor we got was 2, which happens to be $k+1$ for $k=1$.

3.2.2 Severity of Braess paradox in multiple source sink pair networks

The results in this case suggest that the things could be much worse in this case.

It is possible to construct an example in which there are two commodities with delays 0 and 1 and adding a single edge causes the delays to increase to exponential in the graph size. Therefore, adding a single edge to two commodity problem can deteriorate the quality of Nash exponentially. See [LRTW 05] for details and examples.

4 Overview of what is coming next

We will take a break from non-atomic games and consider instead the atomic game (also called discrete games, because the users there are discrete entities). We might come back to non-atomic games later in the course to prove the results stated in Section 3.2.2.

We will also look at use of randomization in games. We will look at randomized Nash equilibria (also called mixed Nash equilibria). More in the next lecture. . .

References

[LRT 04] H. Lin, T. Roughgarden, and E. Tardos, *A Stronger Bound on Braess's Paradox*, SODA 2004, pages 333–334.

[LRTW 05] H. Lin, T. Roughgarden, E. Tardos, and A. Walkover, *Braess's Paradox, Fibonacci Numbers, and Exponential Inapproximability*, in ICALP 2005.

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