

## Price of anarchy for non-atomic selfish routing

We will continue using the following strong assumptions.

- $\ell_e(x)$ , the *delay* on edge  $e$ , is monotone, continuous and differentiable.
- $x \cdot \ell_e(x)$  is convex.

Some of these constraints of  $\ell_e$  can be weakened, but that results only in minor modifications to the proofs.

**Definition:** A flow  $f$  is at Nash equilibrium if for all user types  $i = 1, \dots, k$  and all paths  $P, Q$  from  $s_i$  to  $t_i$  with  $f_P > 0$ , then  $\ell_P(f) \leq \ell_Q(f)$ .

Recall from the previous lecture that

$$\ell_e^*(x) = \ell_e(x) + x \cdot \ell_e'(x)$$

is the *socially aware delay* where you take into account the rate at which you are causing other users pain when you increase the flow on  $e$ .

**Theorem 1** A flow  $f$  minimizes the total delay if for all user types  $i = 1, \dots, k$  and all paths  $P, Q$  from  $s_i$  to  $t_i$  with  $f_P > 0$ , then  $\ell_P^*(f) \leq \ell_Q^*(f)$ . ■

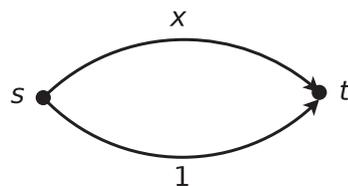


Figure 1: A simple network

What is happening in this very simplistic network in terms of total delay when demand = 1?

- 1) What is the Nash equilibrium?

Apparently, every user picks the upper path. This gives a total delay of 1.

- 2) What is the optimal flow?

That's slightly trickier. We could use brute force, and consider a flow  $f$  such that  $f_{e_1} = \alpha$  and  $f_{e_2} = 1 - \alpha$  (because the demand is 1) where  $e_1$  is the upper edge, and  $e_2$  is the lower one. Then

$$0 = \frac{d}{d\alpha} \text{cost}(f) = \frac{d}{d\alpha} \sum_{e \in E} \ell_e(f_e) f_e = \frac{d}{d\alpha} (\alpha^2 + (1 - \alpha)) = 2\alpha - 1,$$

so  $\alpha = \frac{1}{2}$  where the demand is divided equally between the two paths gives the minimum delay.

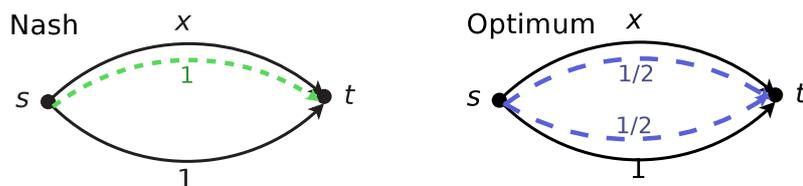


Figure 2: Nash and optimal flows on the simple network

However, theorem (1) gives us an easier method. We need to figure out the  $\ell^*$  delays for the paths.

- For the upper path,  $\ell_{e_1}^*(x) = x + x\ell'_{e_1}(x) = x + x \cdot 1 = 2x$ .
- For the lower path,  $\ell_{e_2}^*(x) = 1 + x\ell'_{e_2}(x) = 1 + x \cdot 0 = 1$ .

The theorem then tells us that the users prefer the upper leg until the  $2x$  becomes secondary to the delay of 1, then they start using the lower one. In other words,  $\ell_{e_1}^*(x) = \ell_{e_2}^*(x)$  which happens when  $x = \frac{1}{2}$ . According to (1), the minimum delay in the network is thus  $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$ .

We have now determined that in this network,

$$\frac{\text{Nash}}{\text{Optimal}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

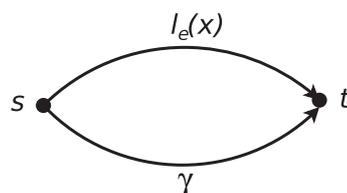


Figure 3: Generalization of the simple network

Now consider the network in figure 3 where  $\gamma$  is a positive constant and the upper and lower edges are named  $e_1$  and  $e_3$ , respectively. Again, how does the Nash equilibrium compare with the optimal solution in terms of total delay?

- 1) What is the Nash equilibrium?

The users keep piling up on the upper leg until the delay reaches  $\gamma$ . Set  $x_N$  as the value where  $\ell_{e_1}(x_N) = \gamma$ . On the upper leg, the Nash flow will be  $\min\{\text{demand}, x_N\}$ , and the remaining demand uses the lower leg. In other words, as long as the demand is less than  $\gamma$ , users will use the upper link, otherwise they use the lower one.

2) What is the optimal flow?

Again, we try to solve for  $\ell_{e_1}^*(x) = \ell_{e_2}^*(x)$  to determine whether we have optimal flow. Now,  $\ell_{e_2}^*(x) = \ell_{e_2}(x) + x \cdot \ell'_{e_2}(x) = \gamma + x \cdot 0 = \gamma$ , so the conditions for theorem (1) can be satisfied. Let  $x_O$  be such that  $\ell_{e_1}^*(x_O) = \gamma$ . The flows in  $e_1$  and  $e_2$  are thus respectively  $\min\{\text{demand}, x_O\}$  and the rest.

"But what about the ratio then?", you may ask. "It must be highly dependent on the delay functions!" As it turns out, however, no matter what valid delay function you pick, as long as  $x \cdot \ell(x)$  is convex, the Nash/Optimal ratio for the two-link networks we just explored is the worst possible.

**Theorem 2** (Roughgarden '02). *In the class of networks that satisfy the conditions, especially the convexity of  $x \cdot \ell(x)$  is convex (includes constant functions), the worst Nash/Optimal ratio arises in a two-link network where one edge has constant delay.*

**Proof:** (Tardos). Let  $G$  be a network where the delay satisfies the conditions. Let  $f$  be a Nash flow in  $G$ , and  $g$  be the optimum flow in  $G$ .

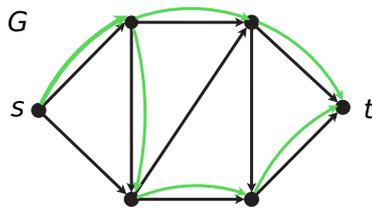


Figure 4: An arbitrary network  $G$ . The green edges denote a Nash flow  $f$ .

We now add a parallel copy  $e'$  of every edge  $e$  in  $G$  and call the resulting graph  $\hat{G}$ . For every edge  $e$  in  $G$ , we set  $\ell_{e'}(x) = \ell_e(f_e)$  in  $\hat{G}$ . Note that  $f_e$  is a constant, so the added edges have constant delay. Let  $\hat{g}$  denote the optimum flow in  $\hat{G}$ .

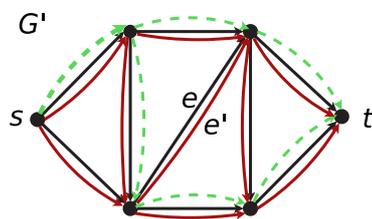


Figure 5: The modified network  $\hat{G}$  with red edges added. The Nash flow stays the same.

Now, observe that even though we added more edges to  $G$ , no user type will care to change. Stated more formally:

**Fact 2.1**  $f$  is a Nash flow in  $\hat{G}$ .

This follows from the fact that we didn't change the shortest paths in  $G$ .

The extra edges, however, can only improve the optimum flow  $\hat{g}$  in  $\hat{G}$  versus  $g$  in  $\hat{G}$ . That is:

**Fact 2.2**  $\text{cost}_{\hat{G}}(\hat{g}) \leq \text{cost}_{\hat{G}}(h)$  where  $h$  is any flow in  $G$ .

Generally speaking, the optimum flow never gets hurt by adding extra options. However, as we saw in the Braess paradox, we can hurt the Nash. Together, facts 1 and 2 say that the Nash/Optimum ratio in  $\hat{G}$  is no better than the ratio in  $G$ . Let's find out why our construction helps the original network.

**Fact 2.3** Minimum total delay flow,  $f^*$ , in  $\hat{G}$  is obtained by optimally rebalancing  $f_e$  between  $e$  and  $e'$  for all  $e$ .

So we will find the optimal flow by local rebalancing of  $f$ , just like we did in our simple network earlier! Let's prove this fact.

**Proof:** Let  $f^*$  be the flow where we have optimally rebalanced  $f_e$  between  $e$  and  $e'$  for all edges  $e$ . We will show that  $f^*$  is the optimal flow. We do that, once again, by taking an arbitrary edge  $e \in G$  and attempt to satisfy the conditions of theorem 1. First, we check if checking whether

$$\ell_e^*(x) = \ell_{e'}(x)$$

for some  $x$ . We have that

$$\ell_{e'}^*(x) = \ell_{e'}(x) + x \cdot \ell_{e'}'(x) = \ell_e(f_e) + x \cdot 0 = \ell_e(f_e)$$

which is constant. Due to our optimal rebalancing,  $f_e^*$  is in fact the  $x$  we were looking for, so it follows that

$$\ell_e^*(f_e^*) = \ell_{e'}^*(f_e^*) = \ell_e(f_e).$$

All flow in the network goes on the shortest path w.r.t.  $\ell$ , so it will also go on the shortest path w.r.t.  $\ell^*$ . The conditions for theorem (1) are thus satisfied, so we can deduce that  $f^*$ , which is the old Nash flow  $f$  with the delay function  $\ell^*$  instead of  $\ell$ , is in fact optimal. ■

We now continue with our proof of the main theorem. From the three facts, we now conclude that

$$\begin{aligned} \frac{\text{Nash}_G}{\text{Optimal}_G} &\leq \frac{\text{Nash}_G}{\text{Optimal}_{G'}} \\ &= \frac{\text{cost}_G(f)}{\text{cost}_{G'}(f^*)} \\ &= \frac{\sum_{e \in E} f_e \ell_e(f_e)}{\sum_{e \in E} (f^*(e) \ell_e(f^*(e)) + f^*(e') \ell_e(f^*(e')))} \\ &\leq \max_{e \in E} \frac{f(e) \ell_e(f(e))}{f^*(e) \ell_e(f^*(e)) + f^*(e') \ell_e(f^*(e'))} \end{aligned}$$

where the last inequality follows from the fact that

$$\frac{a+b}{a'+b'} \leq \max \left\{ \frac{a}{a'}, \frac{b}{b'} \right\}.$$

Now look back at figure 1 and name the two links  $e$  and  $e'$  such that  $e$  takes the maximum and  $f(e)$  is the demand. Then the two-link example actually achieves the aforementioned ratio. ■