CS 684 Algorithmic Game Theory	**Scribe: Thành Nguyen**
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In the lecture on September 7th, we defined a game of network flows, which closely describes the internet traffic. We gave a formula of the sum of all delays which is one of the ways to evaluate the quality of a solution.

$$cost(f) = \sum_i \sum_{P:s_it_ipath} f_P.l_P(f) = \sum_{e\in E} f(e)\cdot l_e(f(e)) \tag{1}$$

Without the proof, we claimed the following theorem:

Theorem 1 *A solution is a Nash if and only if it minimizes the potential function:*

$$\phi = \sum_{e\in E}\int_0^{f(e)} l_e(\xi)d\xi \tag{2}$$

In this lecture using a basic theorem in the theory of convex programming we will prove this theorem and some other results.

Theorem 2 (Convex Programming) *Let $c_e : \mathbb{R} \to \mathbb{R}$ be a continuos, differentiable and convex function for each $e \in E$. Consider the function defined on the set of feasible flows: $C(f) = \sum_{e\in E} c_e(f(e))$, then a flow f minimizes C iff for all i and P, Q s_it_i-paths with $f_P > 0$ then:*

$$\sum_{e\in P} c'_e(f(e)) \le \sum_{e\in Q} c'_e(f(e)) \tag{3}$$

where $c'_e(x)$ is the derivate of function c_e at x.

Comments: We will come back to the proof of this theorem later, but first let's have some comments: The convexity property is critical here to prove the reverse direction: Let's take an example of a simple graph consisting of of two parallel edges g and h, $l_g = 1, l_h = 1 - x^2$, $f(g) = 1, f(h) = 0$, then (3) satisfies but f doesn't minimize C. Another reason is that in many cases (3) only implies that f minimizes C locally.

Proof of Theorem 1: Let $\phi_e(x) = \int_0^x l_e(\xi)d\xi$, and consider ϕ in (2) we have:

$$\phi(f) = \sum_{e\in E} \phi_e(f(e))$$

It's easy to see that ϕ_e is differentiable and $\phi'_e(x) = l_e(x)$ is a monotone increasing function. Thus ϕ_e is a convex function and we can apply Convex Programming theorem. We get : f minimizes ϕ iff

$$\sum_{e\in P} l_e(f(e)) \le \sum_{e\in Q} l_e(f(e))$$

for every i and P, Q path connecting s_i, t_i with $f_P > 0$ and this is exactly the condition of f to be a Nash equilibrium. ■

Other applications: What happens if we apply Convex Programming Theorem for the total delays function (1). We know that if we define $cost_e(x) = x.l_e(x)$, then

$$cost(f) = \sum_{e \in E} cost_e(f(e)).$$

It's natural to assume that $cost_e$ is continuos, differentiable, but what about the convexity? We have:

$$cost'_e(x) = l_e(x) + xl'_e(x)$$

$cost_e$ is convex if l_e is convex, and infact $cost_e$ is convex if l_e is in a much wider class of functions. So it's not very artificial to assume that the total delays function is convex, and thus applying Convex Programming Theorem for the function $cost$ we have the following:

Theorem 3 *Assume $cost_e$ is convex for every $e \in E$ theny f minimizes the total delay function iff $\forall i, P, Qs_it_i$-path and $f_P > 0$ the following satisfies:*

$$\sum_{e \in P}(l_e(f(e)) + f(e)l'_e(f(e))) \leq \sum_{e \in Q}(l_e(f(e)) + f(e)l'_e(f(e))) \tag{4}$$

■

Note the similarity of the Nash condition and the optimility condition for social welfare. The difference is the additional term of $f(e)l'_e(f(e))$. Nash flow simply and selfishly chooses the path with smallest delay, while the socially optimal flow evaluates paths in a social aware way: if additional flow is added to edge e then the $f(e)$ units of flow currently on the edge will see their delay increase at a rate of $l'_e(f(e))$.

There is another way understanding the formula in (4). Let f^* be the flow with minumum total delay, define

$$t_e = f^*(e)l'_e(f^*(e)).$$

This is a fixed number on each edge e. Then in the game where each player tries to minimize the following function selfishly:

$$\sum_{e \in P}(l_e(f(e)) + t_e)$$

the optimum flow with respect to the cost function of total latency (1) is a Nash of this game. In other words one way to make the players play optimally is to put on each edge e an amount of toll t_e and assume that players try to minimize the sum of latency and toll. Formally we have the following theorem:

Theorem 4 *Assuming $cost_e(x) = xl_e(x)$ is a convex function. Consider the game where each player tries to minimize the sum of delay and the total toll function : $\sum_{e \in P} l_e(f(e)) + t_e$, where t_e is the constant defined above using the flow f^* with the minimum total delay, then under this assumption f^* is a Nash equilibrium.* ■