In the lecture on September 7th, we defined a game of network flows, which closely describes the internet traffic. We gave a formula of the sum of all delays which is one of the ways to evaluate the quality of a solution.

\[
\text{cost}(f) = \sum_{i} \sum_{P:s_i t_i \text{ path}} f_P.l_P(f) = \sum_{e \in E} f(e) \cdot l_e(f(e))
\]  

(1)

Without the proof, we claimed the following theorem:

**Theorem 1**  A solution is a Nash if and only if it minimizes the potential function:

\[
\phi = \sum_{e \in E} \int_{0}^{f(e)} l_e(\xi)d\xi
\]

(2)

In this lecture using a basic theorem in the theory of convex programming we will prove this theorem and some other results.

**Theorem 2 (Convex Programming)** Let \( c_e : \mathbb{R} \to \mathbb{R} \) be a continuous, differentiable and convex function for each \( e \in E \). Consider the function defined on the set of feasible flows: \( C(f) = \sum_{e \in E} c_e(f(e)) \), then a flow \( f \) minimizes \( C \) iff for all \( i \) and \( P, Q \) \( s_i t_i \)-paths with \( f_P > 0 \) then:

\[
\sum_{e \in P} c_e'(f(e)) \leq \sum_{e \in Q} c_e'(f(e))
\]

(3)

where \( c_e'(x) \) is the derivative of function \( c_e \) at \( x \).

**Comments:** We will come back to the proof of this theorem later, but first let’s have some comments: The convexity property is critical here to prove the reverse direction: Let’s take an example of a simple graph consisting of of two parallel edges \( g \) and \( h \), \( l_g = 1, l_h = 1 - x^2, f(g) = 1, f(h) = 0 \), then (3) satisfies but \( f \) doesn’t minimize \( C \). Another reason is that in many cases (3) only implies that \( f \) minimizes \( C \) locally.

**Proof of Theorem 1:** Let \( \phi_e(x) = \int_{0}^{x} l_e(\xi)d\xi \), and consider \( \phi \) in (2) we have:

\[
\phi(f) = \sum_{e \in E} \phi_e(f(e))
\]

It’s easy to see that \( \phi_e \) is differentiable and \( \phi_e'(x) = l_e(x) \) is a monotone increasing function. Thus \( \phi_e \) is a convex function and we can apply Convex Programming theorem. We get: \( f \) minimizes \( \phi \) iff

\[
\sum_{e \in P} l_e(f(e)) \leq \sum_{e \in Q} l_e(f(e))
\]

for every \( i \) and \( P, Q \) path connecting \( s_i, t_i \) with \( f_P > 0 \) and this is exactly the condition of \( f \) to be a Nash equilibrium. \[ \blacksquare \]
**Other applications:** What happens if we apply Convex Programming Theorem for the total delays function (1). We know that if we define \( cost_e(x) = x.l_e(x) \), then

\[
  cost(f) = \sum_{e \in E} cost_e(f(e)).
\]

It’s natural to assume that \( cost_e \) is continuous, differentiable, but what about the convexity? We have:

\[
  cost'_e(x) = l_e(x) + x.l'_e(x)
\]

\( cost_e \) is convex if \( l_e \) is convex, and in fact \( cost_e \) is convex if \( l_e \) is in a much wider class of functions. So it’s not very artificial to assume that the total delays function is convex, and thus applying Convex Programming Theorem for the function \( cost \) we have the following:

**Theorem 3** Assume \( cost_e \) is convex for every \( e \in E \) then if \( f \) minimizes the total delay function iff \( \forall i, P, Q, s, t_i \)-path and \( f_P > 0 \) the following satisfies:

\[
  \sum_{e \in P} (l_e(f(e)) + f(e)l'_e(f(e))) \leq \sum_{e \in Q} (l_e(f(e)) + f(e)l'_e(f(e))) \quad (4)
\]

Note the similarity of the Nash condition and the optimality condition for social welfare. The difference is the additional term of \( f(e)l'_e(f(e)) \). Nash flow simply and selfishly chooses the path with smallest delay, while the socially optimal flow evaluates paths in a social aware way: if additional flow is added to edge \( e \) then the \( f(e) \) units of flow currently on the edge will see their delay increase at a rate of \( l'_e(f(e)) \).

There is another way understanding the formula in (4). Let \( f^* \) be the flow with minimum total delay, define

\[
  t_e = f^*(e)l'_e(f^*(e)).
\]

This is a fixed number on each edge \( e \). Then in the game where each player tries to minimize the following function selfishly:

\[
  \sum_{e \in P} (l_e(f(e)) + t_e)
\]

the optimum flow with respect to the cost function of total latency (1) is a Nash of this game. In other words one way to make the players play optimally is to put on each edge \( e \) an amount of toll \( t_e \) and assume that players try to minimize the sum of latency and toll. Formally we have the following theorem:

**Theorem 4** Assuming \( cost_e(x) = x.l_e(x) \) is a convex function. Consider the game where each player tries to minimize the sum of delay and the total toll function : \( \sum_{e \in P} l_e(f(e)) + t_e \), where \( t_e \) is the constant defined above using the flow \( f^* \) with the minimum total delay, then under this assumption \( f^* \) is a Nash equilibrium.