CS 684 Algorithmic Game Theory	**Scribe: Éva Tardos**
Instructor: Eva Tardos	**Friday, August 26, 2005**

Game theory is concerned with situations when users of a system interact, and effect each other. In this class we will be concerned with games that model some issues in computer science. We will start the semester by considering three classes of games: routing, network design, and facility location.

The routing game we will consider has been originally studies in civil engineering as a model of car-traffic. It is also used to model packet traffic on the Internet, but for now let's consider cars. Clearly every morning each car decides which way to drive to work. It is pretty reasonable to assume that cars choose their route the selfishly minimize driving time. However, many cars selecting the same rode causes congestion, and that slows the cars down. This makes it a game: each car effects the driving time of many other cars by its choice of roads. Packets in the Internet don't actually "select" their paths: we will discuss how, and to what extent can this model be viewed as modeling packet-traffic.

Clearly the process that builds and maintains the Internet, and locates certain facilities (such as DNS name servers) is a game in the similar sense. The Internet is built by a set of Independent Network Providers (ISP), where each provider builds its own portion of the network in a selfish fashion, while using the other providers when possible. Looking at the Internet from this perspective, it is magic that it works as well as it does, and we need to strive to understand this better. For network design we will consider highly simplifies games, as so far no unified and convincing model has emerged that helps explain the good behavior of the Internet.

The first question we need to ask is what will happens in such a game. The most natural, and commonly used concept, that we will study in much of this course, is Nash equilibrium. A strategy for each player (say car in the routing game) is a Nash equilibrium if no player can improve his payoff (or value) by changing strategy alone. Note that here define a deterministic, or pure Nash equilibrium, where each player has to choose a single strategy. Later we will also talk about randomized strategies, where players can decide to randomize between strategies. In the first half of the course, we will consider the quality of Nash equilibria, that is, will compare how good are Nash equilibria compared to a centrally built and enforced solution. The ratio of the quality of a Nash equilibrium and a centrally built solution is called the price of anarchy: a price one pays to not have to build and manage a complicated central solution.

The next segment of the source will consider ways to find an equilibrium. It is one of the main open problems in theory to decide if there is a general polynomial time algorithm to find a Nash equilibrium of a game. We will consider special games where such algorithms are known, and try to understand the issues underlying this open problem. A related question one needs to ask if to what extend can be expect players to find an equilibrium, e.g., does natural game play somehow converge to such a solution.

Finally, we will spend time on the general topic of mechanism design: when the price of anarchy is bad, or a Nash equilibrium is hard to find, it would make sense to try to modify the game to make it better for the users. Mechanism design has a large literature, easily can teach a whole course just on this topic. Here we will focus on some issues that relate to the rest of the course, or to some natural computer science topics.

The simplest and maybe most common description or a game uses a game matrix. For example, a two player game where each player has two possible strategies, can be described by a two-by-two matrix, that gives the payoff (the value obtained) by the player. For example, a very simplified version of a goal-kick game involves two player, a goalie and a shooter. Assume the shooter has two option: to shoot left or right, and the goalie has two options: to drive left of right. Now a simple game matrix would look as follows. The shooter chooses a column, and the goalie chooses a row, and the first number in each square is the value for the shooter, while the second number is the value for the goalie.

	L	R
L	$-1, 1$	$1, -1$
R	$1, -1$	$-1, 1$

In this game, there is no pair of fixed strategies that the players can play that neither regrets: the player loosing can always change his strategy and win. There are also game where there are multiple Nash equilibria. Maybe the simplest is the coordination game. Say two players want to attend a game together, either softball or baseball game. They both prefer to be together, but they disagree on which game they prefer. Here is the payoff matrix.

	B	S
B	$2, 5$	$0, 0$
S	$0, 0$	$5, 2$

Now both of them selecting the same game is an equilibrium for both choices. Note that even if the two players share their preference (see below) there still will be two equilibria: attending baseball remains an equilibrium as no player can *alone* change his strategy and improve his outcome.

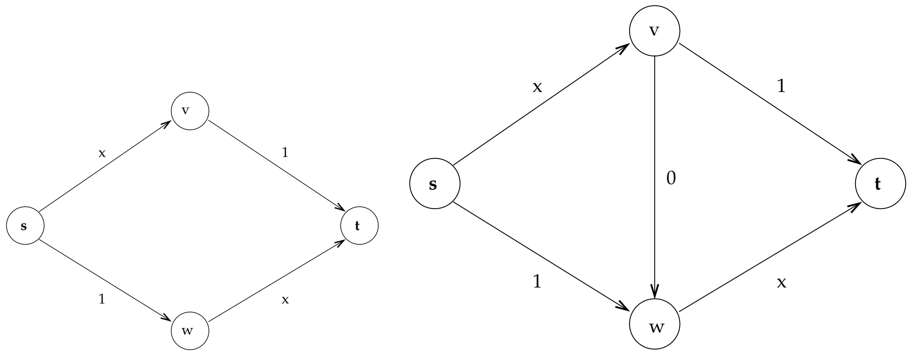
	B	S
B	$2, 2$	$0, 0$
S	$0, 0$	$5, 5$

A similarly simple game that models an economic "pollution game" is defined as follows. Say there are n countries, and each country must decide whether they will have a clean air act. A clean-air act cost them 4 units, but if any country has a clean air act, then all countries gain a value of 3. So for example when there are two countries ($n = 2$) we get the following game matrix.

	P	C
P	$0, 0$	$3, -1$
C	$-1, 3$	$2, 2$

Here is use value 0 to denote the value of no country has a clean-air act. Clearly the best value state is that all countries enact a clean-air act, this results in a value of $3n - 4$ for each of them: a gain of $3n$ from clean air in all n countries, and a cost of 4 for implementing the clean-air act at home.

In the pollution game there is a unique Nash equilibrium, unfortunately, it is for each country to pollute. First, this is an equilibrium, if a single country decides to have a clean air act, all other countries gain 3 in their value, but the country with the clean air act, looses value 1 (cost 4 and gain of 3). Note that no solution, where any country has a clean air act is a Nash equilibrium: each



(a) Before (b) After

Figure 1: Braess's Paradox

country is better of not paying its own cost of 4. Notice that this unique Nash equilibrium is of really bad quality (value 0) compared to the best central solution, of value $3n - 4$ for each player.

As a previous for the next few lectures, we will consider a simple example of the routing game from the beginning of the class, that we will study more extensively during the semester, the Braess paradox.

Consider the simple graph on 4 nodes, s, v, u, t depicted on the right of the figure above. Assume that 1 thousand cars want to drive each morning from s to t. There are 2 roads available: s, v, t and s, u, t. The time (or delay) to drive from s to u or from v to t is 1 hours, independent of traffic, but the road from s to v and from u to t the road is congestion sensitive: if x fraction of the traffic is using this road (that is x times a thousand cars) than it takes x time to get through.

First consider what happens when cars drive on these roads. Assume on day 1 all cars drive $s-v-t$. Now the congestion is 100%, and hence the delay on both edges is 1, and the total delay is 2. We will assume a full information system, so each of the 1000 drives will know that the alternate road $s-u-t$ had no congestion, and hence would have taken them only a bit more than an hour to drive that way (as x there was 0). If on the next day they all switch, than now the alternate path gets congested, and we get oscillation: the two roads will be congested on alternate days. This is an issue we will have to consider when thinking about how to find a Nash equilibrium. For now note that this network has a (unique) equilibrium: namely the cars should divide 1/2-1/2 between the two roads, and all have a delay of 1.5 hours (as x will be 1/2).

To see the Braess paradox, assume a rich donor decides to help, and build a super fast highway: a road v to u with no congestion at all (delay function 0) as shown on the right of the figure. First note that the previous solution is no longer an equilibrium: the cars driving from s towards u having a delay of 1 between these two nodes, will want to take a detour: $s - v - u$ that has current delay of only $1/2 + 0 = 1/2$. Similarly the cars 100% driving from v to t will want to take the detour $v - u - t$, as the drive $v - t$ take 1 time unit, while the detour only takes $1/2 + 0 = 1/2$.

Once everyone switches to the road $s - v - u - t$, the congestion is high everywhere, and the delay is 2. What is worse, this is an equilibrium, despite being all worse off, going back to the previous road does not help the player (it helps a bit the players who do not switch). You may also notice that this is the unique Nash equilibrium in this routing game.

The Braess paradox shows an example when an added edge, that should naturally "improve" the network (or at least not hurt), makes the network worse for selfish traffic. Of course, an added edge cannot hurt the optimum, a centrally designed optimum routing can simply avoid using the edge. This is analogous to the case of the pollution game in that the optimal play is better for everyone than a Nash equilibrium. Notice however that the situation here is not as bad as the pollution game: here the delay goes from 1.5 to 2 (a factor of 4/3 increase) due to selfish behavior. This deterioration is quite limited compared to what we saw in the pollution game. You may wonder how much worse one can make this ratio, e.g. by changing the functions used, the traffic amount, or the network. We'll answer this question later in the course.