Last Time: We began talking about how to remove the cheating assumptions from the Kelley bandwidth sharing game. Following Johari & Tsitsiklis, we set up the game and were in the middle of showing the worst case Opt/Nash ratio.

Today:

- Finish proof from last time about Bandwidth sharing on a single link.
- Begin to Lift this idea to an entire Network

1 Review

There are k players, each with utility $U_i(x)$. This utility is monotonic, concave and continuously differentiable. For convenience, we will assume the utility is strictly monotonic and strictly concave although these requirements are not absolutely necessary. Due to the strict monotonicity assumption, we know that the full Bandwidth is shared. Due to the strict concavity we get uniqueness (see below). Another assumption for convenience is that the capacity of the link, B, equals 1.

FACT: Existence of Optimal Solution.

$\exists$ a unique optimal x_i^* and p^* price such that

- $\sum_i x_i^* =$ B$(= 1)$.
- $U_i'(x_i^*) = p^*$, if $x_i \rangle 0$.
- When $x_i = 0$, then $U_i'(0) \langle p^*$.

The uniqueness is due to the strict concavity assumption we made about the utility function. Also recall that B=1, thus $\sum_i x_i^*$ always equals B and due to our assumption equals 1.

FACT: Existence of Nash Solution.

$\exists$ a Nash $x_1 \ldots x_n$ and a price p such that

- $\sum_i x_i = B(= 1)$.
- $U_i'(x_i)(1 - x_i) = p$ all p, if $x_i \rangle 0$.
- When $x_i = 0$, then $U_i'(0) \langle p$.

Note that this means that users with large utility shade prices. They actually want more, but don't want to pay for it so they shade the prices.

Now that we know an Optimal and Nash solution exist, what is the worst case Optimal/Nash ratio, denoted:

$$max\frac{Optimal}{Nash} \quad = \quad max\frac{\sum_i U_i(x_i^*)}{\sum_i U_i(x_i)}$$

Theorem: [Johari & Tsitsiklis] Given U_i concave, strictly monotone, continuously differentiable, the worst case ratio is:

$$\begin{aligned} max\frac{Optimal}{Nash} &= max\frac{\sum_i U_i(x_i)^*}{\sum_i U_i(x_i)} \\ &\langle \frac{4}{3} \text{ for any Nash } x_i \text{ and Optimal } x_i^*. \end{aligned}$$

Last time we started this proof by showing two steps.
Step 1: It is no loss of generality to assume linear utility, i.e., the worst case occurs when the utilities are of the form $U_i(x) = a_i x + b$.

Step 2: There is no loss of generality to assume utility through origin, that is the worst case occurs when the utilities are of the form $U_i(x) = a_i x$.

2 Continuation of Worst Case Optimal/Nash analysis

2.1 Optimal Solution

We may assume that players are numbered so that 1…k, $a_1 \geq a_2 \geq \ldots \geq a_k$. Now the Optimal solution is easy to see. Give all the bandwidth to Player 1. Player 1 has the largest utility function for all $x \rangle 0$. Optimal Value = a_1 and $x_1^* = 1$. $x_2^* \ldots x_k^* = 0$.

2.2 Nash Solution

Recall from last lecture (and reviewed above) that the Nash solution occurs if there is a price p so that all users get enough bandwidth to satisfy the equation $U_i'(x_i)(1 - x_i) = p$ or have $U_i'(0)\langle p$. In step 2, we assumed utility through origin such that $U_i(x) = a_i x$. Thus $U_i'(x_i) = a_i$. A Nash solution will therefore give bandwidth x_i to player i so that $a_i(1 - x_i) = p$. If $x_i = 0$, then $a_i \leq p$, to see this simply take the above equation and solve for x_i. You end up with $x_i = 1 - \frac{p}{a_i}$. Thus when $a_i \leq p$, i is willing to pay for no positive bandwidth, and will get 0 bandwidth.

2.3 Worst-Case Scenario

Claim: Total utility of Nash is $\geq a_1 x_1 + p(1 - x_1)$, where $p = a_1(1 - x_1)$, and $a_1 \geq a_i$ for all i. This is possible as limit if k goes to ∞.

The equation $p = a_1(1 - x_1)$, comes directly from the Nash property, as we have seen above (recalled from last time). We noticed above that if $x_i \rangle 0$, then $a_i \rangle p$, so the total utility is $\geq a_1 x_1 + p(1 - x_1)$, as claimed.
To see that this is possible in the limit, lets now say that we have lots and lots of people who want Bandwidth $(k\rangle\rangle 0)$. Given enough people taking infinitesimal bandwidth we will use it all.

Thus, let $a_i = p + \epsilon$ for a small enough ϵ, then the equation $a_i(1 - x_i) = p$ determines a very small x_i, and given enough users they use up all the remaining bandwidth of $1 - x_i$.

- The total Nash Utility is:

$$\begin{aligned} F &= a_1 x_1 + (p + \epsilon)(1 - x_1) \qquad (p+\epsilon) \text{ is } a_i \text{ for } i \geq 2 \text{ if } x_i \rangle 0 \\ lim_{\epsilon \to 0} F &= a_1 x_1 + p(1 - x_1) \end{aligned}$$

2.4 Comparison of Nash and Optimum Values

Now, we have bounds for the Optimum and Nash Values and can compare them.

- Optimum Value $= a_1$.
- Nash Value $\geq a_1 x_1 + p(1 - x_1)$
- Also needed is that $a_1(1 - x_1) = p$, from the Nash property.

$$\begin{aligned}
\frac{NashValue}{OptimumValue} &\geq \frac{a_1 x_1 + p(1 - x_1)}{a_1} \\
&\geq \frac{a_1 x_1 + a_1(1 - x_1)(1 - x_1)}{a_1} && p = a_1(1 - x_1) \\
&\geq x_1 + (1 - x_1)^2 \\
&\geq 1 - x_1 + x_1^2 && \text{Minimum when } x_1 = 1/2 \\
&\geq \frac{3}{4}
\end{aligned}$$

Thus, $\frac{OptimumValue}{NashValue} \leq \frac{4}{3}$. Notice that in our worst case example player 1 gets half of the bandwidth in the Nash example and shares the other half with all other users. This is a more "fair" solution, than the optimum, which gives user 1 all the bandwidth.

3 Networks

We will now begin lifting this single-link example to an entire Network. Lets say we have some network. Each edge e has bandwidth b_e. There are k users each with a Path P_i and utility $U_i(x)$. The Paths are fixed and thus not part of the game. Players want end-to-end bandwidth. That is the bandwidth from s_i, the source of P_i, to t_i, the sink of P_i.

- Optimization (Utility Max): $Max \sum_i U_i(x), \qquad x_i \geq 0$
- Bandwidth on edge e not exceeded: $\sum_{i: e \in P_i} x_i \leq b_e$

Theorem: $x_1*, \ldots, x_k^*$ satisfying the above inequalities is optimal (maximum total utility) if and only if exists prices $p_e^* \geq 0$ for $e \in E$ such that

- User i marginal utility for getting more: $U_i'(x_i^*) = \sum_{e \in P_i} p_e^*$ for all i such that $x_i \rangle 0$, and $U_i'(0) \leq \sum_{e \in P_i} p_e^*$ for all i such that $x_i = 0$.
- Underutilized Bandwidth has no price: We have $p_e^* = 0$ for all edges e such that $\sum_{i: e \in P_i} x_i^* \langle b_e$.

Need Linear Programming to prove this. Here are some easy things that one can see directly.

1. All users i: maximize utility subject to prices $\quad Max_{x \geq 0} U_i(x) - x \sum_{e \in P_i} p_e^*$

   Why? To maximize utility, we take derivative, set it to 0 and solve. So we need x_i^* to satisfy $U_i'(x_i^*) - \sum_{e \in P_i} p_e^* = 0$, if $x_i^* \rangle 0$, which is exactly the condition we assumed above.

2. Conditions imply $x_1^* \ldots x_k^*$ Optimal.
Proof Recall that concavity of the utility function implies that $U_i((x) \leq U_i'(x_i^*)(x - x_i^*) + U_i(x_i^*)$, as the right hand side is the tangent line at x_i^*. Using the inequality for all i. Let $x_1, \ldots, x_k$ be another solution. We use this inequality for all i, summing up we get that

$$\sum_i U_i(x) \leq \sum_i [U_i'(x_i^*)(x - x_i^*) + U_i(x_i^*).]$$

Rearranging terms we get that

$$\begin{aligned}
\sum_i U_i(x) - \sum_i U_i(x_i^*) &\leq \sum_i U_i'(x_i^*)(x - x_i^*) \\
&\leq \sum_i (x_i - x_i^*)(\sum_{e \in P_i} p_e^*) \\
&= \sum_e p_e^*(\sum_{i:e \in P_i} x_i - \sum_{i:e \in P_i} x_i^*) \\
&\leq \sum_e p_e^*(b_e - b_e) = 0,
\end{aligned}$$

where the first inequality follows from concavity of the U_i functions, the second inequality follows from the condition on prices (note that when $U_i'(x_i^*)\langle \sum_{e \in P_i} p_e^*$ then we have $x_i^* = 0$, and also $x_i \geq 0$. The last inequality follows from the assumption that $p_e^* \rangle 0$ implies $\sum_{i:e \in P_i} x_i^* = b_e$.