A review on the Kelly bandwidth sharing game

We are dealing with the problem of sharing bandwidth over a single network channel (or edge) with capacity B among k users. We assume every user has a utility function $u_i(x)$ which is continuous, monotonic(increasing), continuously-differentiable and strictly concave.[1] A game approach in solving this problem is

1. The dealer (or owner) proposes a price p per unit of bandwidth.
2. Each user figures out what amount he wants to pay at this price: maximizes $u_i(x) - px$, and announces $w_i = px$ for the optimum value of x.
3. Dealer computes new price $p' = \frac{\sum_i w_i}{B}$ and the amounts that people got at that price $x_i = w_i/p$, and announces these values.
4. If all users are are satisfied (i.e., they got their preferred value on this new price) then terminate, else return to step 2.

In this lecture we answer the following questions:

- Does this game have Nash equilibrium? Is it unique?
- If the Nash exists how good is it?

Defining Nash equilibria

Definition 1 *In the Kelly bandwidth sharing game, $w_1, w_2, \ldots, w_k$ is Nash if for all i the best strategy is w_i, i.e. Player i evaluates $u_i(x_i)$ where $x_i = \frac{B}{\sum_j w_j} w_i$ and $u_i(x_i) - w_i = u_i(\frac{B}{\sum_j w_j} w_i) - w_i$ is maximum.*

We analyze what are the conditions for a Nash equilibrium. Without loss of generality we assume that $B = 1$. [2]. The objective function for the users is

$$\max_{w_i \geq 0} \quad u_i\left(\frac{1}{\sum_j w_j} w_i\right) - w_i$$

This can be obtained by equating the differential with respect to w_i to 0 (keeping the other variables constant). Wherever convenient we replace $\frac{1}{\sum_j w_j} w_i$ by x_i and $\sum_j w_j$ by p

[1] A strictly concave function that is monotonically increasing, is also strictly monotone.

[2] There is only one edge that is shared and hence can be taken as a single unit

$$\frac{d}{dx}\left[u_i\left(\frac{1}{\sum_j w_j} w_i\right) - w_i\right] = 0$$

$$\text{i.e., } \left(\frac{1}{\sum_j w_j} - \frac{w_i}{(\sum_j w_j)^2}\right) u_i'\left(\frac{1}{\sum_j w_j} w_i\right) - 1 = 0$$

$$\text{i.e., } \left(\frac{1}{p} - \frac{x_i}{p}\right) u_i'(x_i) - 1 = 0$$

Therefore, the solution is a Nash if $u_i'(x_i)(1 - x_i) = p$ for users. We notice that this resembles the solution to the original formulation where $u_i'(x_i^*) = p^*$. Taking a step back and looking at the Nash condition, we see that as a user who wants a lot of bandwidth will want bid a low amount to keep the price down.

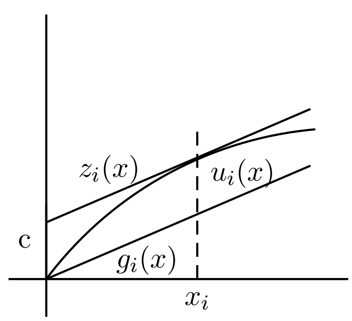
Theorem 1 *For the Kelly bandwidth sharing game, Nash exists and is unique.*

Proof. We give an intuitive argument that it exists. As the dealer we start with a low price and then raise it to meet the bandwidth B. It is unique because we have assumed the functions to be strictly concave and monotone. ■

We now give the price of anarchy. Note that since this is a maximization we have optimum bigger than the Nash.

Theorem 2 (Johari and Tsitsiklis) *The ratio between the optimum utility and Nash is at most* $\frac{4}{3}$.

Proof. The proof is in three steps. In the first two sets we modify the utility function and claim that the Nash solution does not change while the optimum can only increase.



Step 1: Define $z_i(x) = u_i(x_i) + (x - x_i)u_i'(x_i)$ where x_i are the values of the individual bandwidths in the Nash. We use z_i to be the new utility function of all users. i

We notice that the Nash does not change because the condition depends only on the derivative and $z_i'(x_i) = u_i'(x_i)$. Since curve z_i lies above u_i always the optimum could only have increased. Let x_i^* be the optimum solution with utility z_i. Let the value of the objective function for Nash be N and the optimum be O.

Step 2: Define $g_i(x) = u_i'(x_i)x$. Again we have $g_i'(x_i) = u_i'(x_i)$. This ensures that x_i is still a Nash. Further x_j^* is still optimum because $g_i'(x_i) = z_i'(x)$ for all values of x, and optimality condition depends only on $g_i'(x_i^*)$. Let $c = \sum_i z_i(0)$. This is the difference between the value of the objective function with z_i and g_i for both the Nash x_i and the optimum solution x_i^*. Therefore, the new Nash value is $N - c$ and the new optimum value is $O - c$. Since we know $N \leq O$ and $c \leq max\{N, O\}$, we have

$$\frac{O}{N} \leq \frac{O - c}{N - c}$$

The maximum ratio between Optimum and Nash is when the utility functions are of the form $a_i x$ for all i with $a_i \geq 0$. In the next lecture we show that this is at-most $\frac{4}{3}$.

Caveats

Originally, for the game, the utility functions were monotone and strictly concave. In Step 1, we modified the function to z_i, which was a linear function. Thus we lose the properties of having an unique optimum or Nash. This is because the worst case occurs when the functions are not strictly concave. Suppose there were users who had the same utility for an interval of bandwidth, the worst Nash could occur when a lot of bandwidth is given to that user.