

The quality of equilibria in pricing games:

This class, we will deal with quality of routing games when there are prices.

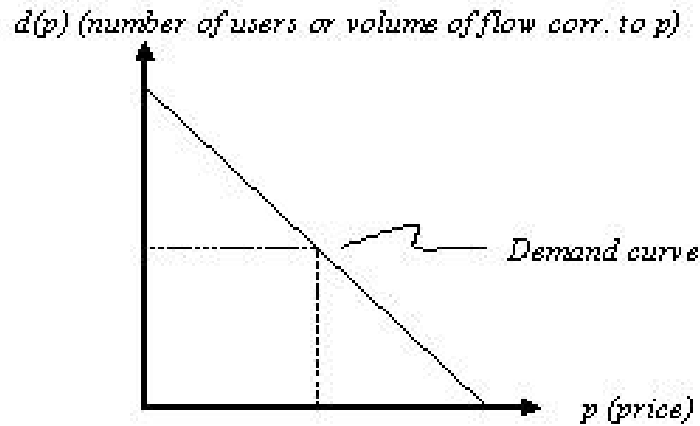
Definition of the game: We have a network with source node s and sink node t connected by a single path of n edges. Each edge is owned by a separate firm who determines the price of the link, i.e. we have n players each controlling one link.



This is a two-stage game:

Stage (Day) 1: At the beginning of the game, each firm i determines the price p_i which he wants to charge for edge i . So a price is determined for the whole path which is the total of prices charged by all firms, i.e. $p = \sum_i p_i$.

Stage (Day) 2: After the prices are announced, users send flow from s to t . The volume of the flow or the number of users (assuming many infinitesimal users) depends on the prices announced. We can interpret as: Each small user has a certain budget and sends flow if the total price do not violate his budget constraint. The relation of flow to total price is given by price-demand curve where x-coordinate represents the total price for whole network and y-coordinate corresponds to the volume of flow that will be send through with this price.

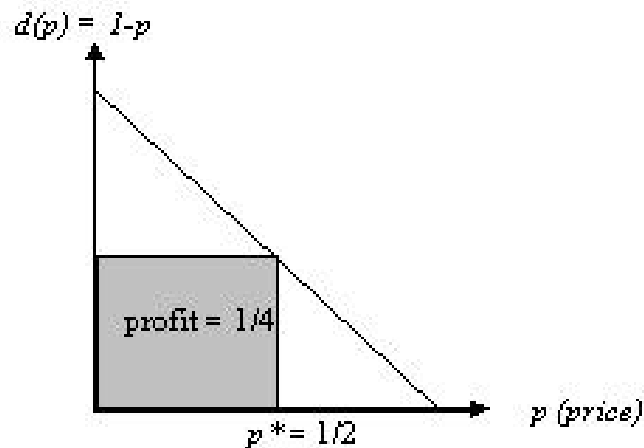


In this game, each firm aims to maximize his profit which can be expressed as $p_i d(p)$.

Note that, the price-demand information is available to all players at the beginning of the game.

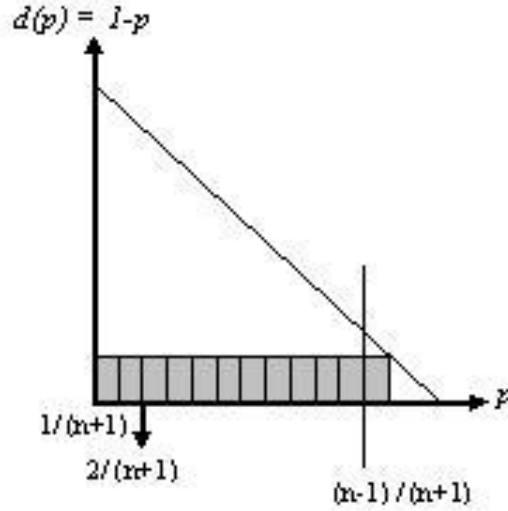
We have two different settings of the game with respect to the competitiveness:

1. Monopolistic Scenario: We have only one firm who controls all links. We will look at the case when we have linear price-demand relation, i.e. $d(p) = 1 - p$



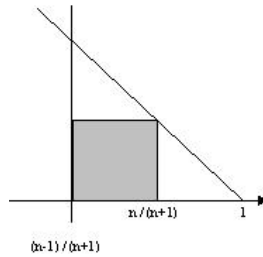
Note that, the area of the shaded square is equal to the profit of the monopolistic firm and it is maximized when $p = 1/4$.

2. Competitive Scenario: We have n distinct firms. When we again consider the linear price-demand relation $d(p) = 1 - p$.



Lemma 1 *The solution with $p_i = 1/(n+1)$, $p = n/(n+1)$, $d(p) = 1/(n+1)$ is a Nash Equilibria.*

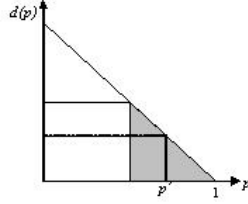
Proof. [by picture] Since all players are symmetric, we can look at the last player (player n). If every other player charges $1/(n+1)$, player n only considers the part of demand curve after $p = (n-1)/(n+1)$. When we zoom into this area, we observe that the game of player n is equal to a monopolistic game in this region, for which we know that player n 's profit is maximized at $p = n/(n+1)$. So, player n does not have any incentive to announce any other price. This is a NE.



Note that, the total profit of the game is $O(1/n)$, which is worse than monopolistic case ($\Omega(1)$). Furthermore, this solution is not socially preferable because every player determines a very high price selfishly resulting in a very low flow on the network.

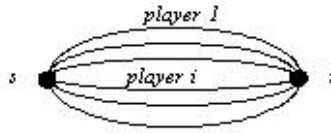
We can define the **social value** of the game as the sum of firm profits and user surplus. User surplus results when there are some players (infinitesimal again) who are willing to pay p' when the

announced total price is $p < p'$. It is the area of the shaded triangle when we are considering the following price-demand curve:



Note that monopolistic scenario achieves higher social value than competitive case, and it is interesting that in the monopolistic case the maximum social welfare is achieved by $p = 0$. We will introduce a new game in which this issue does not arise:

Pricing game: We have a network which consists of a source node s , a sink node t , and n parallel links between source and sink. Each link is controlled by a firm, the price of link i is determined by firm i , and each link i has latency $l_i(x)$. This game is again two-stage:

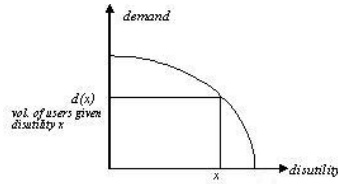


Stage (Day) 1: At the beginning of the game, each firm i determines the price p_i which he wants to charge for edge i as in the previous game.

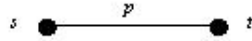
Stage (Day) 2: After the prices are announced, users send flow from s to t . This time, users face with two different sources of unhappiness: price and latency, and there is a tradeoff between this two measures. Although more complex relations can be used, we will use a simple balancing measure for disutility (or unhappiness) which simply adds price and latency, i.e. if a link i has price p_i and latency l_i , then the users using this link will suffer disutility equal to $p_i + l_i$. The demand or number of users willing to send flow on a link depends the disutility and the relation is represented by the demand-disutility curve:

It is easy to observe that disutility is equalized on all used links when the latency function is assumed to continuous and monotonically increasing.

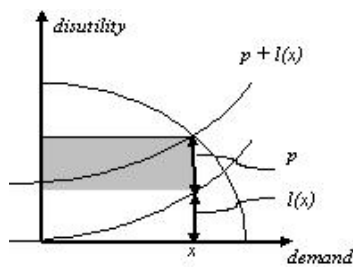
We will look at some examples of this game in order to have some insights:



Example 1: We start with a simple network in which we have only one link so just one player. This player will determine a price p .

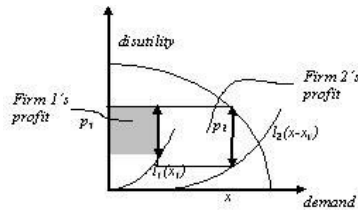
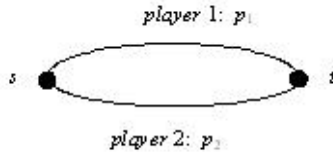


We will make use of a flipped demand-disutility curve for analysis: Note that, given a price we draw the latency function with respect to volume (demand) and shift this curve up with an amount equal to the price. At the point where two curves (shifted latency) and disutility, disutility is equal to latency plus price, so we have a corresponding solution. The area of the shaded rectangle is equal to the profit of the only firm and its goal is to maximize this area.

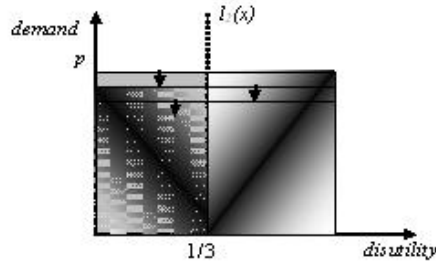


Example 2: In this example, we have two links between the source and demand maintained by two players:

We have common disutility on both links, the area of the vertically shaded region is equal to the profit of first firm whereas the area of the horizontally shaded region is equal to the profit of second firm.



Example 3: This is an example with two links again. We have $l_1(x) = 0$, $l_2(x)=0$ if $x < 1/3$, ∞ o.w. We can see that there is no Nash Equilibria in this game:



Assume at the beginning, player 1 is alone in the game. He will charge as much as possible, i.e. $p_1 = p$ and his profit will be equal to the area of the big shaded square. But, in this case player 2 can charge just below player 1, i.e. $p_2 = p_1 - \epsilon$ and obtain some of the flow. (Player 2's profit if the area on the left shaded diagonally down, whereas Player 1's profit if the area on the right shaded diagonally up). But now, player 1 can decrease a bit more (just below player 2) and all users will use his link. This will continue until we reach to a point where prices are so low that player1 does not care about obtaining all demand but will go for just the demand remaining from player 2 by increasing its price to p again. We cycle from this point with the previous arguments, player 2 increases his prices, etc. So there is no stable point in this game, i.e. NE does not exist.

Note that, although we do not provide the proof, the price of anarchy in the games with linear latency functions or concave demand curve is 1.5.