CS 684 Algorithmic Game Theory	**Scribe: Muthu Venkitasubramaniam**
Instructor: Eva Tardos	**October 26, 2004**

The construction itself is an art, its application to the world an evil parasite.
- L E J Brouwer

Randomized Nash Equilibrium

In this lecture, we discuss the proof of existence of Nash equilibrium in a randomized game. We consider games where the number of players and the strategy sets are finite. We remark here that the finiteness of the strategy set and player set is crucial to the main theorem. We have discussed several games in the past lectures where the players set and the strategy set are finite and infinite, some of which are listed below.

- Job-scheduling game - The number of jobs (i.e. players) is finite, and the number of machines (strategy set) each job could be assigned is finite too.
- Facility-location game - The number of players is finite, and the number of locations where each player could open a facility is finite.
- Job splitting game - The jobs were considered a large number of infinitesimally small jobs, in effect assuming infinite jobs, with finite strategy. There is also a related game with finite jobs with infinite strategies where users control a finite amount of flow, and may split flow.
- Johari Tsitsiklis game - The player set is finite, while the number of strategies is infinite.

Main theorem

There are n players, and each player i has a finite strategy set S_i from which it plays. We consider a valuation function v_i for player i. $v_i(s_1, s_2, \ldots, s_n)$ is the benefit function for player i when the players strategies are $s_1, s_2, \ldots, s_n$ (, i.e. player j plays strategy s_j).

In the games discussed so far, we have considered only pure Nash equilibrium. In a pure Nash, each player plays exactly one strategy from the strategy set. In this lecture, we consider randomized Nash where each player can play several strategies with some probability. Games in general, need not have a pure Nash equilibrium, while randomized Nash equilibrium always exists.

A randomized strategy is a probability distribution $\mathbf{p}_i$ over the strategy set S_i for each player i. $\mathbf{p}_i(s)$ is the probability with which player i plays strategy $s \in S_i$. Since this is a probability distribution,

$$\mathbf{p}_i(s) \geq 0 \text{ and } \sum_{s \in S_i} \mathbf{p}_i(s) = 1.$$

Each player tries to maximize its benefit in expectation. Given a probability distribution $\mathbf{p}_j$ for each player j, the benefit for player i is $E(v_i(s_1, s_2, \ldots, s_n))$. This can be thought of as the benefit player i will get on the average, when the game is played several times with the same randomized strategy.

$$E(v_i(s_1, s_2, \ldots, s_n)) = \sum_{s_i \in S_i, \forall i} \mathbf{p}_1(s_1)\mathbf{p}_2(s_2) \cdots \mathbf{p}_n(s_n) v_i(s_1, s_2, \ldots, s_n)$$

To make our notation cleaner we will use $v_i(p_1, p_2, \ldots, p_n)$ to denote the above expectation. Note that p_i here is a probability distribution for player i.

We now state the main theorem for the lecture,

Theorem 1 (J. F. Nash) *For any game with finite players and finite strategy sets, there exists a randomized Nash equilibrium.*

We take a slight detour into real analysis, to develop machinery to solve the main theorem.

Brouwer's Fixed Point theorem

Theorem 2 (L. E. J. Brouwer) *Every continuous mapping f from a closed, bounded and convex set $C \subset \mathbb{R}^n$ to itself has a fixed point, i.e. there exists $\mathbf{x} \in C$ such that $f(\mathbf{x}) = \mathbf{x}$.*

Refer to Appendix A for the definitions. We do not prove this theorem here, and refer to [2] for the proof.

We given some intuition as to why this is true. Consider continuous functions $f : [0,1] \to [0,1]$. $[0,1]$ is a closed, bounded and convex, and f is continuous. Since these functions satisfy the conditions of the theorem, there exists an $x \in [0,1]$ such that $f(x) = x$. As seen in the Figure 1, $f(0)$ must lie somewhere on the line segment AB and $f(1)$ must lie somewhere on the line segment CD. Since the function is continuous, the path traced by f is a curve from $f(0)$ to $f(1)$. This should meet line segment AD at some point (x' in Figure 1). This point is a fixed point for function f.

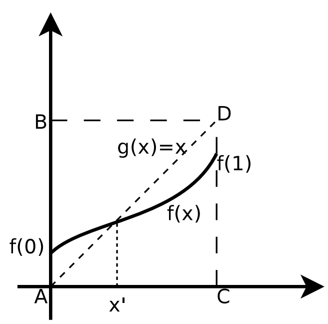


Figure 1: Fixed point in 1-dimension

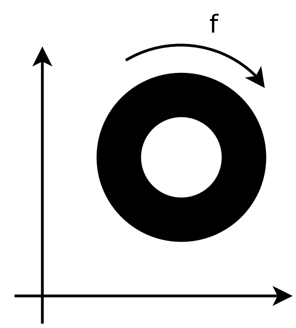


Figure 2: Fixed point in 1-dimension

We illustrate with another example as to why convexity is required. Consider the annulus shown in Figure 2. This is a closed and bounded set in $\mathbb{R}^2$, but not convex. We define a function f on this, that rotates the annulus by some angle. This function is continuous, but does not have a fixed point.

Having set the machinery, we now proceed on to the main theorem.

Proof of Theorem 1: The basic idea is that we invoke the Brouwer's fixed point theorem. The set C under consideration is $\{(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)\}$ where $\mathbf{p}_i$ is the probability distribution over the strategy set S_i. Therefore, $C \subset \mathbb{R}^{|S_1|+|S_2|+\ldots+|S_n|}$. C is closed, bounded and convex, since we have the constraints $\mathbf{p}_i(s) \geq 0$ and $\sum_{s \in S_i} \mathbf{p}_i(s) = 1$.

Given $(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$, define $\mathbf{q}_i$ to be the best response for player i, that is, we define $\mathbf{q}_i$ to be the probability distribution $\mathbf{q}$ maximizing

$$\max_{\mathbf{q}} v_i(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{i-1}, \mathbf{q}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n)).$$

We now define the mapping $f : C \to C$ as follows,

$$f(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n) = (\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n)$$

Clearly, a fixed point for the function f is a Nash equilibrium. If f satisfies the conditions for Brouwer's Theorem, we are done. But we are not quite there, with respect to satisfying the conditions of Brouwer's Theorem for the following reasons:

- The best response $\mathbf{q}_i$ may not be unique for player i. We have seen this in several of our games, such as job scheduling, atomic routing, etc. Hence, f is not well defined.
- Suppose, we chose one of the possible best responses (say the lexicographically first one) as $\mathbf{q}_i$ then f will not be continuous around the points that had multiple best responses.

To resolve these issues, there are two approaches,

1. *Kakutani's Fixed Point Theorem:* This approach deals with more general notions of functions where the function maps the points in the set C to closed and convex subsets of C (in our case the set of best responses), and the theorem states that for such functions there is a point x such that $x \in f(x)$. This is exactly what we need.
2. So we don't have to deal with extending the fixed point theorem, the proof of Kakutani's fixed point theorem, defined in approach 1, for this case we define a point function f which satisfies the conditions of Brouwer's fixed point theorem and hence prove the theorem.

Approach 2

We define $\mathbf{q}_i$ as the probability distribution that maximizes

$$\max_{\mathbf{q}} \left[v_i\left(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{i-1}, \mathbf{q}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n\right) - \|\mathbf{p}_i - \mathbf{q}\|^2 \right]$$

We need to show that,

1. f is a function: We claim that there exists an unique $\mathbf{q}$ that maximizes the right hand side. Intuitively, this is because $\|\mathbf{p}_i - \mathbf{q}_i\|^2$ is strictly concave.
2. f is continuous: With minimal calculus this can be proved.
3. We claim that, if $\mathbf{x} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n)$ is a fixed point, then it is a Nash. To prove this, we assume it is not a Nash and arrive at a contradiction. Suppose, for some player i, there exists a better strategy $\mathbf{q}$ as shown in Figure 3, and suppose the value v_i improved by δ when moving from $\mathbf{p}_i$ to $\mathbf{q}$.

   Now consider moving move along a straight line from $\mathbf{p}_i$ to $\mathbf{q}$. We claim that the function increases

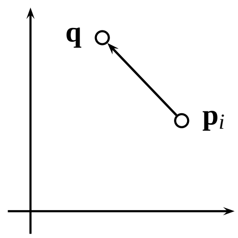
   $$\left[v_i\left(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{i-1}, \mathbf{q}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n\right) - \|\mathbf{p}_i - \mathbf{q}\|^2 \right],$$

   and hence is not maximized at $\mathbf{p}_i$, and hence $(\mathbf{p}_1, \ldots, \mathbf{p}_n)$ is not a fixed point. How do we see that the function decreases?

   - The rate of change on the first term is $\delta / \|\mathbf{p}_i - \mathbf{q}\|$ where $\mathbf{p}_j$ is fixed for $j \neq i$, since the expectation is linear in $\mathbf{q}$.

- On the other hand the initial rate of change in the second term $\|\mathbf{p}_i - \mathbf{q}\|^2$ is 0, as the derivative of x^2 is $2x$ whose derivative at 0 vanishes ($[x^2]'|_{x=0} = 0$).

This implies that there exists some point (in the direction of $\mathbf{q}$ from $\mathbf{p}_i$ possibly nearby $\mathbf{p}_i$) that has a greater value for $v_i(\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_{i-1}, \mathbf{q}, \mathbf{p}_{i+1}, \ldots, \mathbf{p}_n) - \|\mathbf{p}_i - \mathbf{q}\|^2$, thus contradicting the maximality of $\mathbf{q}_i$.



Therefore, by Brouwer's Theorem, we know there exists a fixed point $\mathbf{x}$ and that is a Nash equilibrium.

Finding a Nash

In this lecture, we have shown that finding a Nash equilibrium is the same as finding the fixed point of a function, satisfying conditions of Brouwer's theorem(Brouwer function). It is interesting to note that the converse is also true as proved in [1]. They prove that, given a Brouwer function, there exists a game, whose Nash equilibrium yields a fixed point for the function. In effect, they show that finding a fixed point and finding a Nash has same complexity.

References

[1] C. Daskalakis, P. W. Goldberg, and C. H. Papadimitriou. The complexity of computing a nash equilibrium. Electronic colloquium on computational complexity, 2005. http://eccc.uni-trier.de/eccc-reports/2005/TR05-115/.

[2] James R. Munkres. *Topology*. Prentice-Hall.

A Definitions from Topology

We give some definitions from topology. For this discussion we assume the Euclidean metric space $\mathbb{R}^n$.

Definition 1 *A set $C \subset \mathbb{R}^n$ is closed, if every limit point of an infinite sequence is contained in C.*

A general topological space is defined in terms of its open sets and closed sets are defined as those whose complements are open. In $\mathbb{R}$, finite union of closed intervals form closed sets. In a general topological space, the above definition is called a *complete* set and in the context of the Euclidean space it turns out to be closed too.

Definition 2 *A set $C \subset \mathbb{R}^n$ is bounded, if there exists a real number r such that, for any $\mathbf{x}$ and $\mathbf{y}$ in C, $\|\mathbf{x} - \mathbf{y}\|_2 < r$.*

Definition 3 *A set $C \subset \mathbb{R}^n$ is convex, if for* $\mathbf{x}$ *and* $\mathbf{y}$ *in C, the line joining* $\mathbf{x}$ *and* $\mathbf{y}$ *is contained in C.*

Definition 4 *A mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous if it the inverse image of every open set is an open set.*

Intuitively, this means that for every point $\mathbf{y} = f(\mathbf{x})$ in the range of f, if we take a small (open) ball $\mathbb{B}_y$ covering $\mathbf{y}$, then is an (open) ball $\mathbb{B}_x$ covering $\mathbf{x}$ such that $f(\mathbb{B}_x) \subseteq \mathbb{B}_y$.