Algorithmic Game Theory	**Problem Set 3**
CS 684 Spring 2004	**Due Monday, May 3, 2004**

Solve at least 3 of the following 4 problems. You may solve all 4 for extra credit. The problems are of varying difficulty, but are worth equal credit. Taking scribe notes for class can replace solving one of the problems. We will maintain a FAQ for the problem set on the course Web page. You may use any fact we proved in class without proving the proof or reference.

(1) Consider the following multicast problem on a tree. Given a graph that is a tree $T + (V, E)$ with a root $r \in T$. The nodes are users $v \in V$, and have a private utility u_v for being included in the tree. The goal for this problem is to select a subset $B \subset V - r$ to maximize the "social welfare" $\sum_{v \in B} u_v - c(B)$, where is the cost of the subtree of T spanning nodes $B + r$. Recall that the VCG mechanisms truthfully selects this subset. Give an efficient implementation of the mechanism: show how to compute efficiently the the optimal set, and the prices VCG needs to pay.

(2) Consider the tree problem and notation in the previous question. Now assume that you want to use a cross-monotone cost-sharing method. Recall that the Shapley value is a cross-monotone cost-sharing method.

Given a cost-sharing method ξ and utilities u, we define efficiency loss $\gamma(\xi, u)$ to be the following. Suppose using the cost-sharing ξ will result in accepting the set S for service, and assume B is set maximizing social welfare. Then the efficiency loss is

$$\gamma(\xi, u) = \left(\sum_{v \in B} u_v - c(B) \right) - \left(\sum_{v \in S} u_v - c(S) \right)$$

the difference between the social welfare at S, and the optimal welfare at B. The efficiency loss of the cost-sharing systems ξ is $\gamma(\xi) = \max_u \gamma(\xi, u)$, the efficiency loss for the worst possible utilities.

(a) The Shapley value is often hard to compute, but we claim that in our tree problem it is simple. Give an efficient method to compute the cost shares under Shapley value for the tree problem.
(b) Show that when the graph is a tree T, the cost $c(S)$ is submodular (and hence the Shapley value cost-sharing is cross-monotone).
(c) Show that if the tree consists of a single edge with possibly many users at the one non-root node, then among all cross-monotone cost-sharing methods, the Shapley value cost-sharing has the smallest efficiency loss.
(d) **(optional)** Show that on any tree among all cross-monotone cost-sharing methods, the Shapley value cost-sharing has the smallest efficiency loss.

(3) Consider a submodular and monotone cost function $c(S) \geq 0$ for all subsets $S \subset N$. We want to share the cost $c(N)$ among the set N in some fair way. Let $x_i \geq 0$ for all $i \in N$ be a way to share cost $c(N)$ (that is, $\sum_{i \in N} x_i = c(N)$). Recall the notion on core, that required that $\sum_{i \in S} x_i \leq c(S)$ for all subsets S. Recall the fact that if the cost is submodular, than a cost-sharing in the core exists: we showed that the Shapley value is in the core. We claim that in some sense the Shapley value is not really "fair", and here our goal is to define a more fair cost sharing.

For any vector x, let $x(i)$ denote the sum of the smallest i coordinates (e.g., $x(1)$ is the smallest coordinate in x, and $x(2) - x(1)$ is the second smallest). We say that x is the most egalitarian cost-sharing on the core, if for any other cost-share y in the core $x(i) \geq y(i)$ for all i. For example, this means for $i = 1$ that the smallest cost-share is as high as possible in the most egalitarian vector x. Note that such egalitarian cost-sharing may not exists, even when the core is not empty, as the cost-sharing that maximizes $|x(i)|$ for different integers i may differ.

(a) Show that the Shapley value is not always an egalitarian cost-sharing.

(b) Show that if the cost $c(S) \geq 0$ is monotone increasing and submodular, than there is a most egalitarian cost-sharing in the core. (Hint: you may want to re-read our cost-sharing for the spanning tree game (from April 9th).

(4) Recall the Johari -Tsitsiklis allocation game on a graph $G = (V, E)$ where each player i has a path P_i, and he wants to reserve bandwidth along the edges of P_i. Each edge has a capacity b_e, and each user has a utility function $U_i(x)$, which we assumed is strictly increasing, strictly concave and differentiable.

In the Johari -Tsitsiklis game each player i announced an offered payment $w_i^e \geq 0$ for each of the edges e on its paths (let $w_i^e = 0$ along edges not in the paths). For each edge e we allocate the bandwidth proportionally $x_i^e = b_e w_i^e / (\sum_j w_j^e$, and then each user i can sent $x_i = \min_{e \in P_i} x_i^e$ bandwidth. The users than have a benefit of $U_i(x_i) - \sum_e w_i^e$. We added special rules on allocating bandwidth on edges e with $\sum_e w_i^e = 0$, and showed what a Nash equilibrium is like in this game, and showed that the prize of anarchy in this game is at most 3/4th, that is the welfare $\sum_i U_i(x_i)$ at Nash it at least a 3/4th fraction of the maximum welfare possible.

In defining the Nash equilibrium in this game, we assumed that each player i optimizes his values w_i^e to maximize his personal benefit, given that all the other values w_j^e remain fixed. Here we will assume a bit more flexible users. When player i increases his value w_i^e, then he will get more of the bandwidth of edge e. Now consider another player whose path uses edge e. After ith change the other player j gets less bandwidth on edge e than on his other edges. So naturally, player j will want to reallocate his money to offer more on edge e and maybe less on other edges.

Now define the **Sum Bid** game, in which each player i will only announce a single value w_i. Then the network manager (via a possible a bit complicated method) allocates the money w_i offered by agent i along the edges of path P_i in a pay to make sure that $x_i^e = x_i$ for all edges $e \in P_i$, and allocates the users the resulting bandwidth x_i. For answering this question, you do not have to worry about how the network can do this.

Consider the network of a single paths with k unit capacity edges, and $k + 1$ players, where player i needs capacity on edge i and player the extra player (say player 0) needs capacity on all edges. Assume that the utility of player 0 is $U_0(x) = \gamma x$ for some $\gamma > 0$ and the utility of players $i \geq 1$ is $U_i(x) = x$.

(a) What is the optimal allocation of bandwidth (maximizing total user happiness $\sum_i U_i(x_i)$, and what allocation do we get via the Johari-Tsitsiklis game. What is the worst efficiency loss one can get with this game of $k + 1$ players by varying γ.

(b) Show that the sum-bid game defined above leads to an allocation of $x_0 = \gamma/(\gamma + 1)$ for the user of the whole path, and $x_i = 1/(1 + \gamma)$ for all other users. What is the worst efficiency loss one can get with this game of $k + 1$ players by varying γ.