CS 684 Algorithmic Game Theory	**Scribe: Lucian Leahu**
Instructor: Eva Tardos	**Friday, March 5, 2004**

Market Equilibrium (cont'd)

Last time, we started analyzing equilibrium in markets. We have proved that equilibrium prices exist.

Today, we will consider a slightly different scenario:

- the utility function is linear
- goods are separated from buyers (last time the price was affected by how much money the players were making by selling their goods. Today we assume that there are b amounts of goods on the market. Each player i has m_i money.)

Setup

Consider the case with k buyers and n goods. Let $U_i(x)$ be the utility of buyer i:

$$U_i(x) = U^i \cdot x$$

Recall from last time that each buyer i maximizes $U^i \cdot x$, subject to $p \cdot x \leq m_i$ and $p > 0$. Therefore, the best goods for player i are:

$$\max_j \frac{U^i_j}{p_j}$$

Each player i only wants only wants goods j that are the best given the above definition.

We would like to find a combinatorial/algorithmic way to prove the existence of equilibrium prices.

Consider a bipartite graph consisting of buyers on one side and goods on the other side. We add an edge (i, j) if:

$$\max_l \frac{U^i_l}{p_l} \text{ is achieved for } l = j.$$

Therefore:

- all players' nodes have "outgoing edges" (i.e. there exists a max value for each player)
- it is possible that some nodes representing goods have no "incoming" edges.

Prices p are in equilibrium if for each player i, we have quantity x^i such that:

$$\begin{aligned} x^i_j &> 0 \Rightarrow (i,j) \in E \\ x &\geq 0 \text{ and all products and money are used up} \end{aligned}$$

Given prices, we can decide if they are equilibrium prices using the bipartite graph above and flows:

- buyers have supply m_i
- each good j has demand $\sum_{j=1}^n b_j p_j$ (we want to sell everything)

We will describe an algorithm that monotonically increases prices. The invariant of the algorithm is the fact that prices are set so that in the flow set up, there exists a way to satisfy all demands.

We need to find good starting prices. One idea is

$$p_e = \frac{\min_i m_i}{\sum_j b_j} \forall \text{ good } e$$

It is easy to see that these prices might still not be low enough (some goods may still not be wanted).

So, we set up the graph and if there exists good j that has no edges, lower p_j just until an edge appears. In this way good j has an incoming edge and edges are not removed from the graph.

Claim. The invariant is true at the start.

Proof. Greedy allocation works. (Everything is so cheap, everyone can afford everything). Sell the total quantity of a product to any buyer that wants it.

What goods should *not* raise prices? When increasing all prices by a factor, the ratio $\frac{u^i_j}{p_j}$ stays the same. However, some buyers may run out of money.

We define the *danger set* $S \subseteq$ *goods* such that $\sum_{j \in S} p_j b_j = \sum_{i \in \Gamma(S)} m_i$, where $\Gamma(S) = \{i : \exists j \in S, (i,j) \in E\}$.

Claim. Given j one can find a danger set S containing j via max flow, if one exists.

Algorithm

Repeat until no more danger sets:

1. find a danger set S

2. freeze prices of goods in S
3. remove edges between $\Gamma(S)$ and $\overline{S}$, since buyers in $\Gamma(S)$ are short of money.

No more danger sets $\Rightarrow$ new prices $p_j c$, for $j \notin$ danger set. c is a scalar. (increase prices of products that do not belong to any danger set uniformly)

Raise c until one of the following happens:

- a new edge appears between $\overline{\Gamma}(S)$ and S
- a new node $j \in \overline{S}$ gets into danger set

Need to show:

- new prices satisfy invariant
- there exists a lower bound on the price increase. If all quantities are integers and $u_j^i < U$ then the lower bound would be $\frac{1}{kU^{2k}}$.