1 Ranking

We now study ranking problems. Obviously, a ranking should satisfy the following 2 properties.

**Total** for all $a$ and $b$ in the ranking, either $a > b$ or $b > a$ or $a = b$.

**Transitivity** for all $a$ and $b$ and $c$ in the ranking, if $a < b$ and $b < c$, then $a < c$.

It is easy to see that a simple ordered list satisfies the total ordering and the transitivity properties. But, given a set of item, there are numerous rankings that satisfy the above basic requirement. What are the criteria for the good rankings, then?

Let’s imagine that we have a few number of voters, each of whom produces a ranking for a given set of items, and we want to come up with a single global ranking out of the individual rankings. If our voters have voted the following way,

<table>
<thead>
<tr>
<th>voter 1</th>
<th>voter 2</th>
<th>voter 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
</tr>
<tr>
<td>b</td>
<td>c</td>
<td>a</td>
</tr>
<tr>
<td>c</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

what is the good global ordering for such votes? Voter 1 and 3 want $a > b$, voter 1 and 2 want $b > c$, and voter 2 and 3 want $c > a$. However, there is no global ordering that satisfies all three of $a > b$, $b > c$ and $c > a$, as this violates the transitivity.

2 Arrow’s theorem

Arrow studied the problem of coming up with a global ranking from individual rankings.

He assumed three axioms that a reasonable global ranking should satisfy and then showed that there is no global ranking that can satisfy all three axioms. His axioms are the followings.

**Axiom 1. non-dictator**: The algorithm cannot let the global ranking be identical with a single voter’s ranking

**Axiom 2. unanimity**: If everyone prefers $a$ to $b$, the global ranking should prefer $a$ to $b$.

**Axiom 3. independence of irrelevant alternatives**: If individuals modify their rankings but keep the order of $a$ and $b$ the same, then the global ranking should not change its order of $a$ and $b$.

Arrow has proved that there is no global ranking that satisfies all three axioms.

**Theorem** Any algorithm for creating a global ranking that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

To prove the above theorem, we first prove the following lemma which will be used in the proof of the theorem.

**Lemma** If an element $b$ appears in extreme position (either first or last) in each individual ranking, then
The global ranking that satisfying the axioms should also place \( b \) in either first or last.

<table>
<thead>
<tr>
<th>voter 1</th>
<th>voter 2</th>
<th>( \ldots )</th>
<th>global ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>( b )</td>
<td>( \ldots )</td>
<td>( ? )</td>
</tr>
</tbody>
</table>

**Proof** Suppose to the contrary, \( b \) is not first nor last in the global ranking. Then, \( \exists a \) and \( c \) such that \( a > b > c \). By transitivity, \( a > c \).

Now, let all voters move \( c \) above \( a \) in their individual rankings. By unanimity, the global ranking must have \( c > a \). The relative order of \( b \) and \( a \) is the same for each voter as \( b \) is in the extreme position, thus, by independence of irrelevant alternatives, the global ranking of \( b \) and \( a \) does not change. The same argument holds for \( b \) and \( c \) that the global ranking of \( b \) and \( c \) does not change. By transitivity, \( a > c \). This is a contradiction.

Given the Lemma above, now we can prove Arrow’s Theorem.

**Proof.** Consider a set of ranking where every voter ranks \( b \) last, thus by unanimity, in the global ranking, \( b \) should be the last.

Let voters one by one move \( b \) to the first rank, again by unanimity, in the end of this process, \( b \) should be the first in global ranking. As proved above that when \( b \) is either the last or the first in individual voter’s ranking, \( b \) must be either the last or the first in global ranking. Therefore, there must be a voter \( v \) where global rank of \( b \) jumps from the last to the first.

We now argue that \( v \) is a dictator.

First we will show that, \( v \) is a dictator for all \( a \) and \( c \), not involving \( b \).

For any pair elements other than \( b \) in \( v \)’s ranking, we denote the higher ranked one as \( a \) and the lower one as \( c \), thus, \( a > c \).

Let’s denote the system before \( b \) is moved from the last to the first in \( v \) as in State I, which is illustrated as below:

<table>
<thead>
<tr>
<th>voter1</th>
<th>voter2</th>
<th>( \ldots )</th>
<th>voter( v )</th>
<th>( \ldots )</th>
<th>voter( n-1 )</th>
<th>voter( n )</th>
<th>global ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b )</td>
<td>( b )</td>
<td>( \ldots )</td>
<td>:</td>
<td>:</td>
<td>( \ldots )</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>( \ldots )</td>
<td>( a )</td>
<td>( \ldots )</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>:</td>
<td>:</td>
<td>( \ldots )</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>( \ldots )</td>
<td>( c )</td>
<td>( \ldots )</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>( \ldots )</td>
<td>:</td>
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<td>( \ldots )</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>( \ldots )</td>
<td>( b )</td>
<td>( \ldots )</td>
<td>( b )</td>
<td>( b )</td>
<td>( b )</td>
</tr>
</tbody>
</table>

State I: before \( b \) is moved to the first in \( v \).

As we proved before, when \( b \) is moved to the top in \( v \), the system enters State II where \( b \) jumps to the first in the global ranking, as shown below:

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State II: after $b$ is moved to the first in $v$.

Now, let $v$ modify his ranking by moving $a$ above $b$, so that in $v$, $a > b > c$. As illustrated in the figure below, we call the system at this moment in State III.

The global ranking places $a$ in front of $b$, because the order of $a$ and $b$, should stay the same as in State I. More specifically, as there is no other voter changing his rank of $(a,b)$ during this period from State I to State III, by the independence of irreleavant alternative, it keeps $a > b$ in State III, for the global ranking.

On the other hand, in the global ranking, we know that $b > c$ in State II because $b$ is globally on the top; and since there is no voter changing his order of $(b,c)$ in between State II and State III, the global rank of $(b,c)$ stays the same as $b > c$ in State III.

By transitivity, the global ranking must put $a > c$ in State III, which follows the order of $(a,c)$ in $v$’s rank.

Similarly, we can show that if we put $v$’s rank of $c$ in front of $b$ after State II, globally, $c > a$ in State III, which again follows the order of $(a,c)$ in $v$’s rank.

Also, in State III, no matter how other voters change their orders of $(a,c)$, the relative positions between of $(a,b)$ and $(b,c)$ are not going to change (because $b$ is at extreme positions in other voters’ rankings), thus the global ranking of $(a,c)$ stays the same as in $v$’s ranking.

Hence we can say that $v$ is a dictator over every pair $(a,c)$, when $a \neq b$ and $c \neq b$.

Now let’s consider another element $c$. By placing $c$ at the bottom of each individual rank and moving $c$ to the first one by one, we can find a voter $v_c$, whose change of $c$’s position brings $c$ to the top in global ranking. Repeat the same process as before, the system again will go through State I, II, III and we can prove that $v_c$ is a dictator over every pair $(a,b)$ not involving $c$.

We claim that $v = v_c$, because the global ranking has to agree with $v$ by $a$, $c$ and $v_c$ by $a$, $b$, $v$ and $v_c$ must be the same. More details of this proof will be provided in next lecture.

\[\square\]