CS 683: Advanced Design & Analysis of Algorithms	April 2, 2008

Lecture 28

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1 Ranking

We now study ranking problems. Obviously, a ranking should satisfy the following 2 properties.

Total for all a and b in the ranking, either $a > b$ or $b > a$ or $a = b$.

Transitivity for all a and b and c in the ranking, if $a < b$ and $b < c$, then $a < c$.

It is easy to see that a simple ordered list satisfies the total ordering and the transitivity properties.

But, given a set of item, there are numerous rankings that satisfy the above basic requirement. What are the criteria for the good rankings, then?

Let's imagine that we have a few number of voters, each of whom produces a ranking for a given set of items, and we want to come up with a single global ranking out of the individual rankings. If our voters have voted the following way,

voter 1	voter 2	voter 3
a	b	c
b	c	a
c	a	b

what is the good global ordering for such votes? Voter 1 and 3 want $a > b$, voter 1 and 2 want $b > c$, and voter 2 and 3 want $c > a$. However, there is no global ordering that satisfies all three of $a > b$, $b > c$ and $c > a$, as this violates the transitivity.

2 Arrow's theorem

Arrow studied the problem of coming up with a global ranking from individual rankings.

He assumed three axioms that a reasonable global ranking should satisfy and then showed that there is no global ranking that can satisfy all three axioms. His axioms are the followings.

Axiom 1. non-dictator *: The algorithm cannot let the global ranking be identical with a single voter's ranking*

Axiom 2. unanimity *: If everyone prefers a to b, the global ranking should prefer a to b.*

Axiom 3. independence of irrelevant alternatives *: If individuals modify their rankings but keep the order of a and b the same, then the global ranking should not change its order of a and b.*

Arrow has proved that there is no global ranking that satisfies all three axioms.

Theorem *Any algorithm for creating a global ranking that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.*

To prove the above theorem, we first prove the following lemma which will be used in the proof of the theorem.

Lemma *If an element b appears in extreme position (either first or last) in each individual ranking, then* *the global ranking that satisfying the axioms should also place b in either first or last.*

voter 1	voter 2		...		global ranking
b	b				
$\vdots$	$\vdots$	$\vdots$	...	$\vdots$	?
		b		b	

Proof Suppose to the contrary, b is not first nor last in the global ranking. Then, $\exists a$ and c such that $a > b > c$. By transitivity, $a > c$.

Now, let all voters move c above a in their individual rankings. By unanimity, the global ranking must have $c > a$. The relative order of b and a is the same for each voter as b is in the extreme position, thus, by independence of irrelevant alternatives, the global ranking of b and a does not change. The same argument holds for b and c that the global ranking of b and c does not change. By transitivity, $a > c$. This is a contradiction.

Given the Lemma above, now we can prove Arrow's Theorem.

Proof. Consider a set of ranking where every voter ranks b last, thus by unanimity, in the global ranking, b should be the last.

Let voters one by one move b to the first rank, again by unanimity, in the end of this process, b should be the first in global ranking. As proved above that when b is either the last or the first in individual voter's ranking, b must be either the last or the first in global ranking. Therefore, there must be a voter v where global rank of b jumps from the last to the first.

We now argue that v is a dictator.

First we will show that, v is a dictator for all a and c, not involving b.

For any pair elements other than b in v's ranking, we denote the higher ranked one as a and the lower one as c, thus, $a > c$.

Let's denote the system before b is moved from the last to the first in v as in State I, which is illustracted as below:

$voter_1$	$voter_2$	$\cdots$	$voter_v$	$\cdots$	$voter_{n-1}$	$voter_n$	global ranking
b	b	$\cdots$	$\vdots$	$\cdots$			
$\vdots$	$\vdots$	$\cdots$	a	$\cdots$	$\vdots$	$\vdots$	$\vdots$
		$\cdots$	$\vdots$	$\cdots$			
$\vdots$	$\vdots$	$\cdots$	c	$\cdots$	$\vdots$	$\vdots$	$\vdots$
		$\cdots$	$\vdots$	$\cdots$			
$\vdots$	$\vdots$	$\cdots$	b	$\cdots$	b	b	b

State I: before b is moved to the first in v.

As we proved before, when b is moved to the top in v, the system enters State II where b jumps to the first in the global ranking, as shown below:

$voter_1$	$voter_2$	$\cdots$	$voter_v$	$\cdots$	$voter_{n-1}$	$voter_n$	global ranking
b	b	$\cdots$	b	$\cdots$			b
		$\cdots$	$\vdots$	$\cdots$			$\vdots$
$\vdots$	$\vdots$	$\cdots$	a	$\cdots$	$\vdots$	$\vdots$	a
		$\cdots$	$\vdots$	$\cdots$			$\vdots$
$\vdots$	$\vdots$	$\cdots$	c	$\cdots$	$\vdots$	$\vdots$	c
		$\cdots$	$\vdots$	$\cdots$	b	b	$\vdots$

State II: after b is moved to the first in v.

Now, let v modify his ranking by moving a above b, so that in v, $a > b > c$. As illustrated in the figure below, we call the system at this moment in State III.

$voter_1$	$voter_2$	$\cdots$	$voter_v$	$\cdots$	$voter_{n-1}$	$voter_n$	global ranking
b	b	$\cdots$	a	$\cdots$			a
		$\cdots$	b	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$			b
		$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\cdots$	c	$\cdots$			c
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	b	b	$\vdots$

State III: after a is moved in front of b in v.

The gloabal ranking places a in front of b, because the order of a and b, should stay the same as in State I. More specifically, as there is no other voter changing his rank of (a,b) during this period from State I to State III, by the independence of irrelavant alternative, it keeps $a > b$ in State III, for the global ranking.

On the other hand, in the global ranking, we know that $b > c$ in State II because b is globally on the top; and since there is no voter changing his order of (b,c) in between State II and State III, the global rank of (b,c) stays the same as $b > c$ in State III.

By transitivity, the global ranking must put $a > c$ in State III, which follows the order of (a,c) in v's rank.

Similarly, we can show that if we put v's rank of c in front of b after State II, globally, $c > a$ in State III, which again follows the order of (a,c) in v's rank.

Also, in State III, no matter how other voters change their orders of (a,c), the relative positions between of (a,b) and (b,c) are not going to change (because b is at extreme positions in other voters' rankings), thus the global ranking of (a,c) stays the same as in v's ranking.

Hence we can say that v is a dictator over every pair (a,c), when $a \neq b$ and $c \neq b$.

Now let's consider another element c. By placing c at the bottom of each individual rank and moving c to the first one by one, we can find a voter v_c whose change of c's position brings c to the top in global ranking. Repeat the same process as befoer, the system again will go through State I, II, III and we can prove that v_c is a dictator over every pair (a,b) not involving c.

We claim that $v = v_c$, because the global ranking has to agree with v by a, c and v_c by a, b, v and v_c must be the same. More details of this proof will be provided in next lecture.

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