CS 683 Advanced Design and Analysis of Algorithms February 27, 2008

Lecture 17: Random Walks

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Random Walks

Walks In 1-dimension

Let X_i correspond to the direction of movement at time step i. That is, if at time i in our random walk we move right, $X_i = 1$; if instead we moved left, $X_i = -1$. Let S_i be the location at time i. Then, our location at time n is:

$$S_n = X_1 + \cdots + X_n \tag{1}$$

Let z_i be the probability that $S_i = 0$ and let f_i be the probability that the first return to the origin is at time i. Then for some k:

$$z_{2k} = f_0\, z_{2k} + f_2\, z_{2k-2} + f_4\, z_{2k-4} + \cdots + f_{2k}\, z_0 \tag{2}$$

where $f_0 = 0$ and $z_0 = 1$. Using this notation, we define the generating functions for z and f, respectively, as:

$$z(x) = \sum_{m=0}^{\infty} z_{2m}\, x^m \tag{3}$$

$$f(x) = \sum_{m=0}^{\infty} f_{2m}\, x^m \tag{4}$$

Claim: $z(x) = 1 + z(x)\, f(x)$
Proof:

$$\begin{aligned} z(x) &= 1 + z(x)f(x) \\ &= \underbrace{1 + z_0 f_0}_{z_0} + \underbrace{(z_0 f_2 + z_2 f_0)}_{z_2}\, x + \underbrace{(z_0 f_4 + z_2 f_2 + z_4 f_0)}_{z_4}\, x^2 \\ &= z_0 + z_2\, x + z_4\, x^2 + \cdots \\ &= z(x) \end{aligned} \tag{5}$$

Claim: $z(x) = \frac{1}{\sqrt{1-x}}$
Proof:

$$\begin{aligned} z(x) &= \sum_{m=0}^{\infty} z_{2m} \cdot x^m \\ &= \sum_{m=0}^{\infty} \binom{2m}{m} \left(\frac{1}{2}\right)^{2m} \cdot x^m \\ &= \frac{1}{\sqrt{1-x}} \quad , \qquad \text{by Binomial Theorem} \end{aligned} \tag{6}$$

Claim: $f(x) = 1 - \sqrt{1-x}$
Proof:

$$\begin{aligned} f(x) &= \frac{z(x)-1}{z(x)} \quad , \qquad \text{by (5)} \\ &= 1 - \frac{1}{z(x)} \\ &= 1 - \sqrt{1-x} \quad , \qquad \text{by (6)} \end{aligned} \tag{7}$$

Now it's easy to see that $f(1) = \Pr(\text{return to origin}) = 1$. Thus the probability that a random walk in 1-dimension will return to the origin is 1.[1]

Undirected Graphs

For arbitrary undirected graphs, we can analyze random walks by analyzing similar electrical networks. Consider Figure 1.

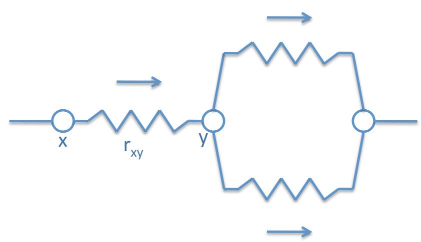

Figure 1: A simple electrical network.

In this case, r_{xy} is the resistance between nodes x and y. Now consider Figure 2.

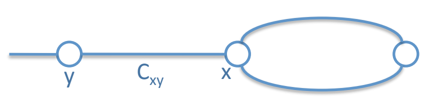

Figure 2: A simple electrical network.

C_{xy} is the conductance between nodes x and y and corresponds to the inverse of the resistance, *i.e.* $C_{xy} = \frac{1}{r_{xy}}$. We will say that the probability of traveling from x to y in our random walk is:

$$P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} = \frac{C_{xy}}{C_x} \tag{8}$$

Definition: We will say that a graph is periodic if the greatest common divisor (g.c.d.) of all cycles in the graph is greater than 1. A graph is aperiodic if it is not periodic.

Theorem: If a graph is aperiodic, then a random walk on that graph will converge to a stationary probability, *i.e.* each node will have some fixed proportion of the time spent in the walk. We will

[1] The same is true for 2-dimensions. However, for 3-dimensions the probability is ≈ 0.65.

use f_x to refer to the stationary probability of a node x.

Claim: $f_x = \frac{C_x}{\sum_y C_y} = \frac{C_x}{C_{eff}}$
Proof:

$$\begin{aligned} f_x &= \sum_y f_y P_{yx} \\ &= \sum_y \frac{C_y}{C_{eff}} \frac{C_{yx}}{C_y} \\ &= \sum_y \frac{C_{yx}}{C_{eff}} \\ &= \frac{C_x}{C_{eff}} \end{aligned} \tag{9}$$

Suppose each edge had resistance 1 in the electrical network like in Figure 3.

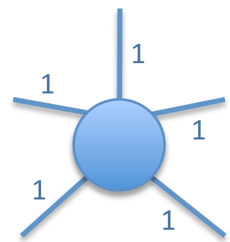

Figure 3: A node with uniform resistance edges.

Then the probability of taking any edge is $\frac{1}{\deg(x)}$, $C_x = \deg(x)$, $C = 2m$ and $f_x = \frac{deg(x)}{2m}$, where m is the number of edges.

Harmonic Functions

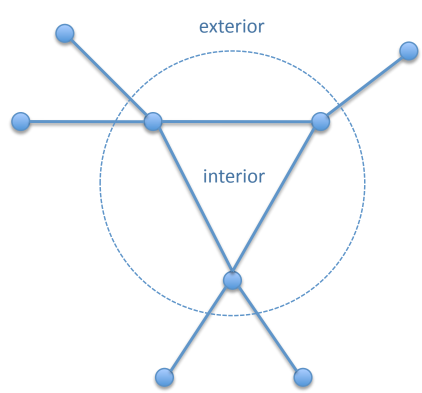

Figure 4: Values at interior vertices is some weighted function of adjacent vertices.

A harmonic function is a function on vertices, where values at an interior vertex is some weighted function of adjacent vertices (see Figure 4).

Some useful features of harmonic functions include:

- There exists a unique harmonic function for any given set of boundary values.
- If g and h satisfy weight sums, then so does $g-h$. Furthermore, the resulting boundary nodes have the value 0.
- Harmonic functions take on their minimum and maximum values on the boundary.

More Electrical Networks

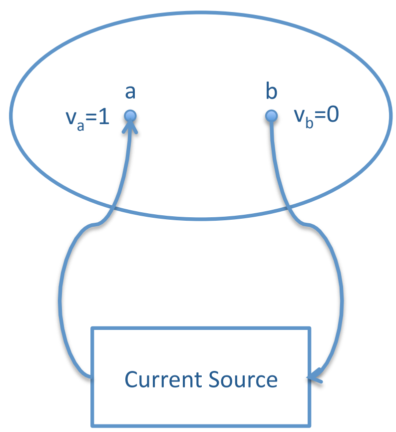

Figure 5: A simple electrical network.

Choose two vertices a and b, *e.g.* Figure 5, and attach a current source. Adjust the current so that $v_a = 1$ in reference to $v_b = 0$. Induce a current in each edge and a voltage at each vertex. Then, the voltage at each vertex is the probability of a random walk starting at that vertex and reaching a before reaching b. The current flowing through each edge is the net number of traversals of that edge in one random walk from a to b.