Random Walks

Walks In 1-dimension

Let $X_i$ correspond to the direction of movement at time step $i$. That is, if at time $i$ in our random walk we move right, $X_i = 1$; if instead we moved left, $X_i = -1$. Let $S_i$ be the location at time $i$. Then, our location at time $n$ is:

$$S_n = X_1 + \cdots + X_n \quad (1)$$

Let $z_i$ be the probability that $S_i = 0$ and let $f_i$ be the probability that the first return to the origin is at time $i$. Then for some $k$:

$$z_{2k} = f_0 z_{2k} + f_2 z_{2k-2} + f_4 z_{2k-4} + \cdots + f_{2k} z_0 \quad (2)$$

where $f_0 = 0$ and $z_0 = 1$. Using this notation, we define the generating functions for $z$ and $f$, respectively, as:

$$z(x) = \sum_{m=0}^{\infty} z_{2m} x^m \quad (3)$$

$$f(x) = \sum_{m=0}^{\infty} f_{2m} x^m \quad (4)$$

Claim: $z(x) = 1 + z(x) f(x)$

Proof:

$$z(x) = 1 + z(x) f(x)$$

$$= \frac{1}{1-z_0} \left( z_0 f_0 + (z_0 f_2 + z_2 f_0) x + (z_0 f_4 + z_2 f_2 + z_4 f_0) x^2 \right)$$

$$= z_0 + z_2 x + z_4 x^2 + \cdots$$

$$= z(x) \quad (5)$$

Claim: $z(x) = \frac{1}{\sqrt{1-x}}$

Proof:

$$z(x) = \sum_{m=0}^{\infty} z_{2m} \cdot x^m$$

$$= \sum_{m=0}^{\infty} \left( \begin{array}{c} 2m \\ m \end{array} \right) \left( \frac{1}{2} \right)^{2m} \cdot x^m$$

$$= \frac{1}{\sqrt{1-x}} \quad , \quad \text{by Binomial Theorem} \quad (6)$$
Claim: \( f(x) = 1 - \sqrt{1 - x} \)

Proof:

\[
f(x) = \frac{z(x) - 1}{z(x)}, \quad \text{by (5)}
\]

\[
= 1 - \frac{1}{z(x)}
\]

\[
= 1 - \sqrt{1 - x}, \quad \text{by (6)} \tag{7}
\]

Now it’s easy to see that \( f(1) = \Pr(\text{return to origin}) = 1 \). Thus the probability that a random walk in 1-dimension will return to the origin is \( 1 \).\(^1\)

Undirected Graphs

For arbitrary undirected graphs, we can analyze random walks by analyzing similar electrical networks. Consider Figure 1.

![Figure 1: A simple electrical network.](image)

In this case, \( r_{xy} \) is the resistance between nodes \( x \) and \( y \). Now consider Figure 2.

![Figure 2: A simple electrical network.](image)

\( C_{xy} \) is the conductance between nodes \( x \) and \( y \) and corresponds to the inverse of the resistance, \( i.e. \), \( C_{xy} = \frac{1}{r_{xy}} \). We will say that the probability of traveling from \( x \) to \( y \) in our random walk is:

\[
P_{xy} = \frac{C_{xy}}{\sum_z C_{xz}} = \frac{C_{xy}}{C_x} \tag{8}
\]

Definition: We will say that a graph is periodic if the greatest common divisor (g.c.d.) of all cycles in the graph is greater than 1. A graph is aperiodic if it is not periodic.

Theorem: If a graph is aperiodic, then a random walk on that graph will converge to a stationary probability, \( i.e. \), each node will have some fixed proportion of the time spent in the walk. We will

\(^1\)The same is true for 2-dimensions. However, for 3-dimensions the probability is \( \approx 0.65 \).
use $f_x$ to refer to the stationary probability of a node $x$.

Claim: $f_x = \frac{C_x}{\sum_y C_y} = \frac{C_x}{C_{eff}}$

Proof:

$$f_x = \sum_y f_y P_{yx} = \sum_y \frac{C_y C_{yx}}{C_{eff} C_y} = \sum_y \frac{C_{yx}}{C_{eff}} = \frac{C_x}{C_{eff}}$$ (9)

Suppose each edge had resistance 1 in the electrical network like in Figure 3.

![Figure 3: A node with uniform resistance edges.](image)

Then the probability of taking any edge is $\frac{1}{\deg(x)}$, $C_x = \deg(x)$, $C = 2m$ and $f_x = \frac{\deg(x)}{2m}$, where $m$ is the number of edges.

**Harmonic Functions**

![Figure 4: Values at interior vertices is some weighted function of adjacent vertices.](image)
A harmonic function is a function on vertices, where values at an interior vertex is some weighted function of adjacent vertices (see Figure 4).

Some useful features of harmonic functions include:

- There exists a unique harmonic function for any given set of boundary values.
- If $g$ and $h$ satisfy weight sums, then so does $g - h$. Furthermore, the resulting boundary nodes have the value 0.
- Harmonic functions take on their minimum and maximum values on the boundary.

More Electrical Networks

Choose two vertices $a$ and $b$, e.g. Figure 5, and attach a current source. Adjust the current so that $v_a = 1$ in reference to $v_b = 0$. Induce a current in each edge and a voltage at each vertex. Then, the voltage at each vertex is the probability of a random walk starting at that vertex and reaching $a$ before reaching $b$. The current flowing through each edge is the net number of traversals of that edge in one random walk from $a$ to $b$. 

Figure 5: A simple electrical network.