CS 683 Advanced Design and Analysis of Algorithms February 8, 2008

Lecture 9: More on Generating Functions

Instructor: John Hopcroft *Scribe: Yao Yue*

Revisiting Monotone Property

Define *monotone property* in terms of adding edges to $G(n,p)$. If a property, once occurred in a graph, is kept while adding subsequent edge(s), then this property is called a monotone property of $G(n,p)$.
Now consider the following statement: if $p_2 > p_1$, then $Prob(G(n,p_2) \in Q) > Prob(G(n,p_1) \in Q$, where Q is the set of graphs having a given property. Can you prove that the above definition yields this property? In other words, can you prove the statement is always true for a monotone property?
Further, can you show that this statement yields monotone property? (If so, it is an equivalent definition of monotone property.) Give a proof or suggest a counterexample.

Generating Functions

GF for Fibonacci Sequence (continued)

Generating functions are often used for sequences defined by recurrence relations. Fibonacci sequence $f_0, f_1, f_2, \ldots$ is an example of this kind. It satisfies $f_0 = 0, f_1 = 1, f_i = f_{i-1} + f_{i-2} (i \geq 2)$. Generation function of Fibonacci sequence is as follows (see previous lecture notes):

$$\begin{aligned} f(x) &= \frac{x}{1-x-x^2} = \frac{\frac{\sqrt{5}}{5}}{1-\phi_1 x} + \frac{-\frac{\sqrt{5}}{5}}{1-\phi_2 x} \\ &= \frac{\sqrt{5}}{5}\left[1 + \phi_1 x + \phi_1^2 x^2 + \cdots\right] - \frac{\sqrt{5}}{5}\left[1 + \phi_2 x + \phi_2^2 x^2 + \cdots\right] \end{aligned}$$

where $\phi_1 = \frac{1+\sqrt{5}}{2}, \phi_2 = \frac{1-\sqrt{5}}{2}$. Note that $\phi_1 > 1, -1 < \phi_2 < 0$.
The coefficient of x^n is f_n, using the property that $|\phi_2| < 1$ we have

$$f_n = \frac{\sqrt{5}}{5}\phi_1^n - \frac{\sqrt{5}}{5}\phi_2^n \simeq \frac{\sqrt{5}}{5}\phi_1^n$$

when n is large. Since f_n is an integer, we further have

$$\begin{aligned} f_n &= \frac{\sqrt{5}}{5}\phi_1^n - \frac{\sqrt{5}}{5}\phi_2^n \\ &= \lfloor \frac{\sqrt{5}}{5}\phi_1^n - \frac{\sqrt{5}}{5}\phi_2^n \rfloor \\ &= \lfloor \frac{\sqrt{5}}{5}\phi_1^n - \frac{\sqrt{5}}{5}\phi_2^n + \frac{\sqrt{5}}{5}\phi_2^n \rfloor \\ &= \lfloor \frac{\sqrt{5}}{5}\phi_1^n \rfloor \end{aligned}$$

GF for Branching Process

A branching process start with a root, a probability distribution for the children $p_0, p_1, p_2, \ldots$ are the probability of a node having corresponding number of children. Every node has the same probability distribution. The generation function for the roof node is

$$f(x) = \sum_{i=0}^{\infty} p_i x^i$$

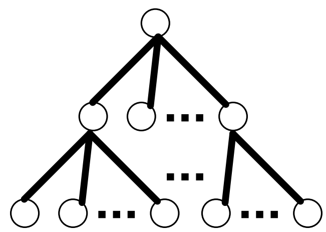

Figure 1: A Branching Process

Prove that the generating function for children in j-th generation $f_j(x)$ is $f_j = f_{j-1}(f(x)), j \geq 2$.
Proof We first state the following two facts:
1) If $x_1, x_2, \ldots, x_n$ are independent and identically-distributed random variables, generating function for the sum of $x_1 + x_2 + x_3 + \ldots + x_n$ is $f^n(x)$
(GF of $x_1 + x_2$ is $f^2(x) = p_0^2 + (p_0p_1 + p_1p_0)x + (p_0p_2 + p_1p_1 + p_2p_0)x^2 + \cdots$, where $f(x) = \sum_{i=0}^{\infty} p_i x^i$. Extend it to the general case by induction.)
2) $f_j(x) = c_0 + c_1x^1 + \ldots + c_ix^i + \cdots$, c_i is the probability of having exactly i children in the j-th generation. Figure 2 shows the relation between numbers of nodes in the j-th and $j+1$-th generation.

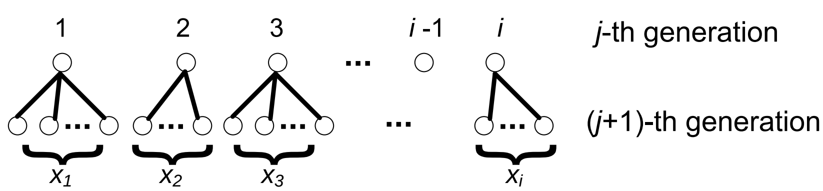

Figure 2: Branching from the j-th to $j+1$-th generation

Given i children in the j-th generation, by fact 2) we know the generating function for the number of children in $j+1$-th generation is $f^i(x)$. Since the probability of i children in the j-th generation is c_i, the generating function of the number of children in the $j+1$-th generation is $\sum_{i=0}^{\infty} c_i f^i(x)$. Compare generating functions of the j-th and $j+1$-th generation, it is easily noticed that $f_{j+1}(x) = \sum_{i=0}^{\infty} c_i f^i(x) = f_j(f(x))$. Therefore, $f_j(x) = f_{j-1}(x) = \underbrace{f \cdot f \cdots f}_{j}(x)$.

In the generating function sequence $f(x), f_2(x), f_3(x), \ldots$, the coefficients of $f_j(x)$ are all positive. For $x \in [0,1]$, the derivative of $f_j(x)$ is always greater than 0. Hence $f_j(x)$ is monotonously increasing and convex on interval $[0,1]$ if $a_0 < 1$ (otherwise $f_j(x)$ is trivially 1).
Consider the roots of equation $f(x) = x$, with above properties of $f(x)$, only one of the roots falls between [0,1]. Let m be the slope of $f(x)$ at $x = 1$. If:

$m < 1$, branching process dies out with probability 1

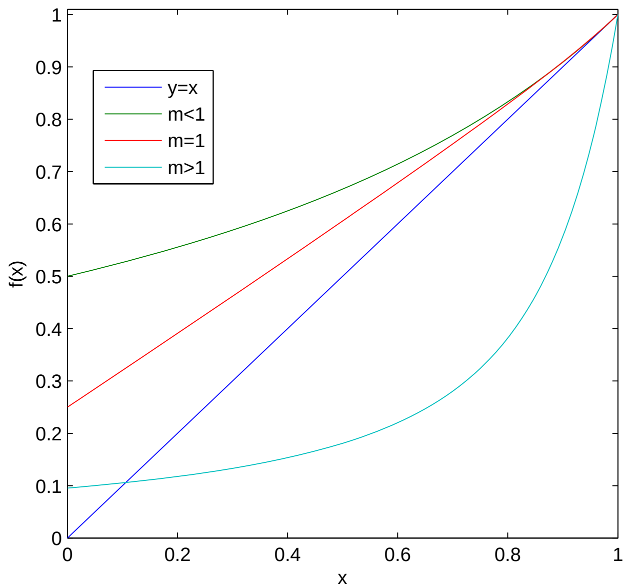

Figure 3: Conditioning on slope at $x = 1$

$m = 1,$ it further depends on p_1

$p_1 = 1,$ $f(x)$ degenerates to $f(x) = x$

$p_1 < 1,$ branching process dies out with probability 1

$m > 1,$ branching process dies out with probability q, where q is the root of $f(x) = x$ that falls within $(0, 1)$.

Exercise Prove that $f_j(x)$ converges to q for $x \in (0, 1)$.