CS683: calculating the effective resistances

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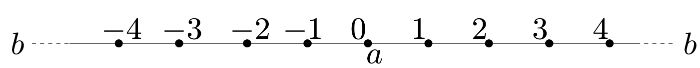
On February 29th we saw that, given a graph in which each edge is labelled with a resistance (analogy with an electrical circuit), and given two vertices of this graph a and b, the probability of reaching b before returning to a (considering a path that starts from a) is $P_{\text{escape}} = \frac{C_{\text{eff}}}{C_a} = \frac{R_a}{R_{eff}}$. The purpose of this lecture will be to calculate C_{eff}, and thus P_{escape}, in some special cases. We are mainly interested in calculating the probability of escape from the center of a volume, first in one dimension, then in 2 dimension and 3 dimensions. In all the lecture we will keep the notation a for our starting node.

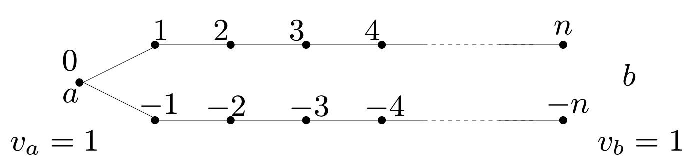
Throughout, $C_a = \sum_x C_{ax}$, where $C_{ax} = \frac{1}{R_{ax}}$ and x is a neighbor of a in the graph.

1 In one dimension

In one dimension our volume looks like:



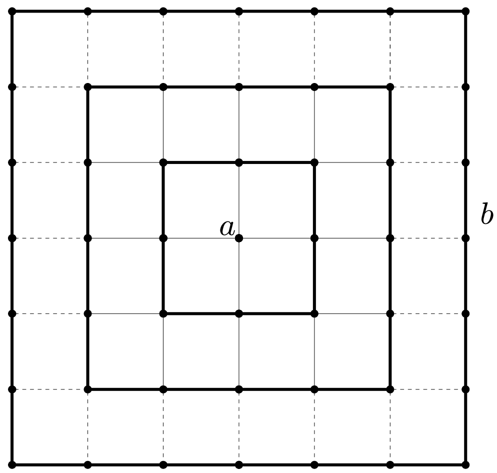
Bending in half, this can also be viewed as:



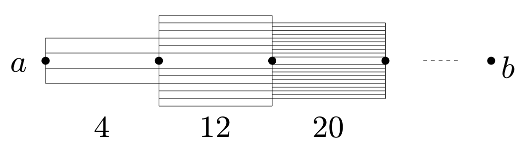
Therefore $R_{\text{eff}} = \frac{n+1}{2}$, so $P_{\text{escape}} = \frac{C_{eff}}{C_a} = \frac{1}{C_a R_{\text{eff}}} = \frac{1}{2R_{\text{eff}}} = \frac{1}{n+1} \longrightarrow 0$ when $n \to +\infty$.

2 In two dimensions

Here also, we want to calculate the resistance. Here we want a lower bound on the resistance, so we put some resistances to zero (the ones that are bold in the scheme and correspond to 'serial edges'); the effect of doing this is to reduce R_{eff} and give us an upper bound on P_{escape}:



This can also be viewed as:

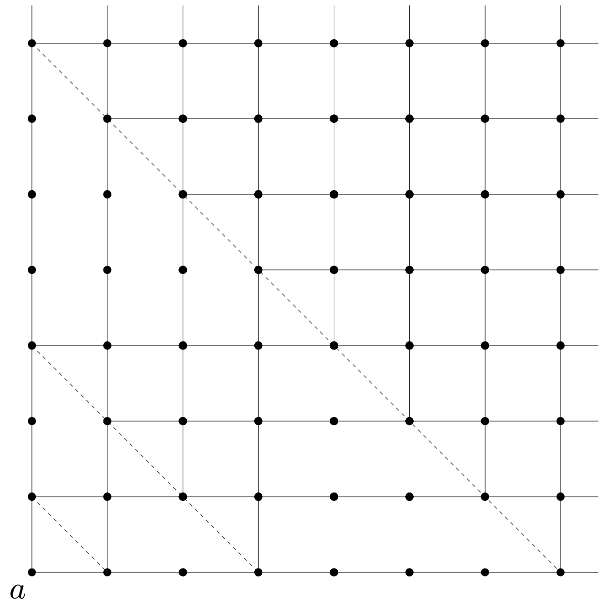


Therefore $R_{\text{eff}} \geq \frac{1}{4} + \frac{1}{12} + \frac{1}{20} + \cdots = \frac{1}{4}\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = \frac{1}{4}\sum_{i=1}^{\inf} \frac{1}{2*i-1} = \Theta(\log n)$. Therefore the probability of escaping $0 \leq P_{\text{escape}} \leq \frac{1}{4\Theta(\log n)}$ and so $P_{escape} \to 0$ when $n \to \infty$.

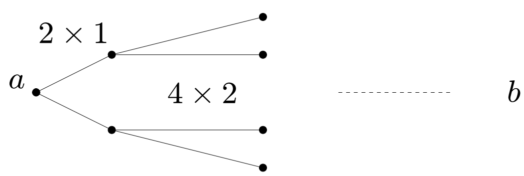
3 In three dimensions

3.1 Lower bound on P_{escape}

Let us present the technique in 2 dimensions, then apply it to the 3-dimensional case. As before, to get a lower bound on P_{escape}, we need an upper bound on R_{eff}, therefore we will put some resistances to infinity, i.e., just cut the wires(on 'parallel edges'). We will do that in the following way: first draw in dashes the lines of equation $x+y = 2^n - 1$ (this will be $x+y+z = 2^n - 1$ in 3 dimensions); then from the origin, follow a path up and a path right until hitting a dotted line; when hitting a dotted line, split each path reaching the dotted line into a path up and a path right again, etc. Doing that, all the resistances that are not on a path are considered to be $+\infty$, and the others are considered to be 1. Here is a picture illustrating this process:



Now, this can also be viewed as(p, r), where p is the number of paths and r is the resistance of each path:



Now, since the resistance of each path between dotted lines grows at a rate of 2^i and the number of parallel paths between dotted lines also grows at a rate of 2^i, these factors balance each other and therefore we would have $R_{\text{eff}} \leq \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + \cdots$.

But we are really in 3 dimensions, therefore we get that the number of paths grows faster than 2^i, infact it grows at the rate of 3^i, while the length of each path grows at the same rate:

$$
\begin{aligned}
R_{\text{eff}} &\le \frac{1}{3} + 2\times\frac{1}{9} + 4\times\frac{1}{27} + \cdots \\
&\le \frac{1}{3}\sum_{k=0}^{+\infty}\left(\frac{2}{3}\right)^k \\
&\le \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) \\
R_{\text{eff}} &\le 1
\end{aligned}
$$

Now for P_{escape}

$$
P_{escape} = \frac{1}{C_a R_{eff}} P_{\text{escape}} \ge \frac{1}{C_a \times 1}
$$

$$
P_{\text{escape}} \ge \frac{1}{3}
$$

3.2 Upper bound on P_{escape}

Now to get P_{escape}'s upper bound we find a lower bound on R_{eff} using the same principles as with the 2D lower bound, ie using cubes instead of squares:

$$
\begin{aligned}
R_{eff} &\ge \frac{1}{6} + \frac{1}{6*9} + \frac{1}{6*25} + \cdots + \frac{1}{6*numberofnodesonsideofacube} + \cdots \\
R_{\text{eff}} &\ge \frac{1}{6}\left(1 + \frac{1}{9} + \frac{1}{25} + \cdots\right) \\
&\ge \frac{1}{6}\left(\left(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36}\cdots\right) - \left(\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right)\right) \\
&\ge \frac{1}{6}\left(\left(1 - \frac{1}{4}\right)\left(1 + \frac{1}{4} + \frac{1}{9} + \cdots\right)\right) \\
R_{\text{eff}} &\ge \frac{\pi^2}{16}
\end{aligned}
$$

Hence getting to the upper bound on P_{escape}:

$$
P_{escape} \ge \frac{R_a}{R_{eff}}
$$

$$
P_{escape} \ge \frac{8}{3\pi^2}
$$

4 Page Rank

We now briefly discuss special graphs with respect to page rank and leave detailed discussion for the next lecture.

When crawling the web it is possible to land a node, or group of nodes, such that there are no out edges. This presents a problem when ending a crawl or in unduly landing at certain nodes too often for page rank to be accurate. An effective way around this is to randomly restart a web crawl, for some reason restarting with probability 0.15 works well.

Consider the case in which a web crawl from a page restarts with probability 0.15, leaves the page altogether with probability 0.425 and follows a self-loop with probability 0.425. Allowing that the page's rank is x, the page rank will eventually converge to the following formula:

$$x = 1 + 0.425xx = 1.74$$

Even if the a person seeks to artifically increase their page rank by adding an infinite number of self loops the page rank will still converge to every out possibility going to the page and with probability 0.15 the web crawl will leave giving:

$$x = 1 + 0.85xx = 6.6666\cdots$$