CS 683: Advanced Design and Analysis of Algorithms	March 5, 2008
Lecture Notes 20	
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1 Random Walks on Undirected Graphs

Definition. Given vertices u and v in a graph, *hitting time* h_{uv} is the expected number of moves in random walk starting at u to reach v.

The effect of adding an edge is dependent on u and v.

Lemma. If u and v are connected by an edge, then

$$h_{uv} + h_{vu} \leq 2m$$

where m is the number of edges in the graph.

Hitting time is not symmetric. That is

$$h_{uv} \neq h_{vu}$$

Consider the following counterexample in figure 1 involving a graph with n vertices.

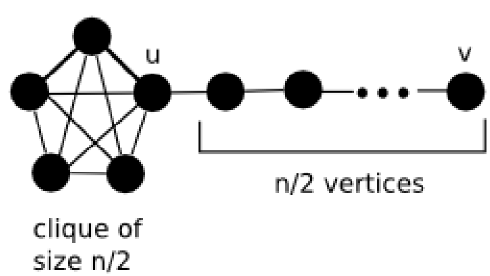

Figure 1: Diagram of graph that is a counterexample to the claim that hitting time is symmetric

It can be shown that

$$h_{vu} = \Theta(n^2) \quad (1)$$
$$h_{uv} = \Theta(n^3) \quad (2)$$

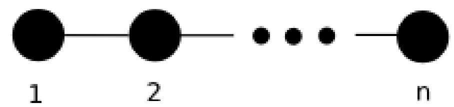

Figure 2: The graph of the solved exercise

A different solved example is depicted in figure 2.

Find an explicit formula for $h_{1,n}$.

The first step in solving this problem is to find a recurrence relation for hitting time between two adjacent vertices. Starting at vertex $1 < i < n$, the random walk has an equal probability of going to vertex $i+1$ and $i-1$. If the random walk were to move to $i+1$, the time to hit vertex $i+1$ is 1. If the random walk were to move to $i-1$, the walk would first have to move back to vertex i before it could hit vertex $i+1$. Therefore, the hitting time if the walk were to first move to $i-1$ would be $1 + h_{i-1,i} + h_{i,i+1}$.

The hitting time overall would be given by the recurrence

$$h_{i,i+1} = \frac{1}{2} \times 1 + \frac{1}{2}(1 + h_{i-1,i} + h_{i,i+1}) \quad (3)$$

$$h_{i,i+1} = 2 + h_{i-1,i} \quad (4)$$

Note that $h_{i,i+1} = 2i - 1$ solves this recurrence.

Then, the hitting time $h_{1,n}$ would be

$$\sum_{i=1}^{n-1} 2i - 1 = (n-1)^2$$

Other measures of time in random walks:

Definition. *Communte time* C_{uv} is the expected number of steps for a walk starting at u to reach v at least once and then return to u.

Definition. *Return time* is the expected number of steps for a walk starting at a vertex to return to that vertex after leaving it.

Definition. *Cover time* C_u is the expected number of steps for a walk starting at u to reach every vertex in the graph of which u is a member.

Definition. The *cover time* of a graph G is the maximum cover time C_u when considering all vertices u in G.

Adding edges to a graph may increase or decrease the cover time of its vertices.

Consider a complete graph. The cover time of the graph is $O(nlogn)$, where there are n vertices in the graph.

Starting at any vertex v in the graph, the probability of going from v to any particular vertex other than v is $\frac{1}{n-1}$. The probability of not going to a particular vertex is $1 - \frac{1}{n-1}$. The probability of not reaching a vertex after m moves in a random walk is $(1-\frac{1}{n-1})^m$. The probability of reaching a particular vertex at least once after m moves is $1-(1-\frac{1}{n-1})^m$. Finally, the probability of reaching all vertices after m moves is $(1-(1-\frac{1}{n-1})^m)^n$.

Proof. The cover time of a complete graph is in $O(nlogn)$, where n is the number of vertices.
If we suspect that the expected value of $m \varepsilon O(nlogn)$ for a complete graph, we should substitute $nlogn$ for m and check for a threshold to see if our suspicions are correct. We then proceed to take the limit as n approaches infinity.

$$\lim_{n\to\infty}(1-(1-\frac{1}{n-1})^{nlogn})^n = \tag{5}$$

$$\lim_{n\to\infty}(1-e^{-logn})^n = \tag{6}$$

$$\lim_{n\to\infty}(1-\frac{1}{n})^n = \tag{7}$$

$$\frac{1}{e} \tag{8}$$

Since the limit is a non-zero constant, $nlogn$ is the location of a threshold. Therefore, the cover time of a complete graph is indeed in $O(nlogn)$.