Random Graphs

1. \( G(n,p) \) : a random graph of \( n \) vertices and edge present with probability \( p \).

2. Degree distribution

\[
P(\text{deg} = k) = \binom{n}{k} p^k (1 - p)^{n-k}
\]

, where \( n \) is the number of vertices and \( p \) is edge probability.

Ex.1 \( G(n, \frac{1}{2}) \)

\[
P(\text{deg} = k) = \binom{n}{k} \frac{1}{2^n}
\]

This satisfies binomial distribution with mean \( \frac{n}{2} \).

However, above doesn’t satisfy for the cases of \( p = \frac{d}{n} \), where \( d \) is constant.

Instead, degree distribution now satisfies Poisson distribution,

Proof Let’s hold \( d = np \), where \( d \) is constant. Thus,

\[
P(\text{deg} = k) = \lim_{n \to \infty} \binom{n}{k} \left( \frac{d}{n} \right)^k \left( 1 - \frac{d}{n} \right)^{n-k}
\]

For \( n \to \infty \),

\[
n - k \approx n \quad , \quad \binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \approx \frac{n^k}{k!}
\]

Hence,

\[
P(\text{deg} = k) \approx \lim_{n \to \infty} \frac{n^k d^k}{k!} \left( 1 - \frac{d}{n} \right)^n = \frac{d^k}{k!} e^{-d}
\]

The degree distribution is Poisson with coefficient \( d \).

Ex. 2 \( G(n, \frac{1}{n}) \)

\[E(\text{deg}) = 1\]

Highest degree is \( \frac{\log(n)}{\log(\log(n))} \).

What is the probability of seeing a vertex of degree \( \frac{\log(n)}{\log(\log(n))} \) ?

Solution Let \( k = \frac{\log(n)}{\log(\log(n))} \). We know \( \log(k^k) = k \cdot \log(k) \).

Hence,

\[
\log(k^k) = \frac{\log(n)}{\log(\log(n))} \left( \log(n) - \log(\log(n)) \right) \approx \log(n)
\]

Hence, \( k^k \approx n \), \( k! \leq k^k \approx n \).
\[ P(\text{deg} = k) \geq \frac{1}{n} e^{-1} \]

Therefore,
\[
P\left(\text{no vertex has degree } \frac{\log(n)}{\log\log(n)}\right) \leq \left(1 - \frac{1}{n} e^{-1}\right)^n
\]
\[
P\left(\text{some vertex has degree } \frac{\log(n)}{\log\log(n)}\right) \geq 1 - \left(1 - \frac{1}{n} e^{-1}\right)^n
\]
\[\approx 1 - e^{-\frac{1}{e}} \approx 0.31\]