1/21/2008 CS683 Lecture1 Haibo Lu

Random Graphs

1. G(n,p) : a random graph of n vertices and edge present with probability p

2. Degree distribution

$$P(\deg = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

, where n is the number of vertices and p is edge probability.

Ex.1 $G(n, \frac{1}{2})$

$$P(\deg = k) = \binom{n}{k} \frac{1}{2^n}$$

This satisfies binomial distribution with mean $\frac{n}{2}$.

However, above doesn't satisfy for the cases of $p = \frac{d}{n}$, where d is constant. Instead, degree distribution now satisfies Poisson distribution,

<u>Proof</u> Let's hold d=np, where d is constant. Thus,

$$P(deg = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{d}{n}\right)^k (1 - \frac{d}{n})^{n-k}$$

For $n \to \infty$,

$$n - k \approx n \quad , \quad \binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!} \approx \frac{n^k}{k!}$$

Hence,

$$P(deg = k) \approx \lim_{n\to\infty} \frac{n^k}{k!}\frac{d^k}{n^k}(1 - \frac{d}{n})^n = \frac{d^k}{k!}e^{-d}$$

The degree distribution is Poisson with coefficient d.

Ex. 2 $G(n, \frac{1}{n})$

$E(\deg) = 1$

Highest degree is $\frac{\log(n)}{\log\log(n)}$

What is the probability of seeing a vertex of degree $\frac{\log(n)}{\log\log(n)}$?

<u>Solution</u> Let $k = \frac{\log(n)}{\log\log(n)}$. We know $\log(k^k) = k * \log(k)$. Hence,

$$\log(k^k) = \frac{\log(n)}{\log\log(n)}\left(\log\log(n) - \log\log\log(n)\right) \approx \log(n)$$

Hence, $k^k \approx n$, $k! \leq k^k \approx n$

$$P(deg = k) \geq \frac{1}{n} e^{-1}$$

Therefore,

$$P\left(no\ vertex\ has\ degree\ \frac{\log(\mathrm{n})}{\mathrm{loglog}(\mathrm{n})}\right) \leq (1 - \frac{1}{\mathrm{n}}\mathrm{e}^{-1})^{\mathrm{n}}$$

$$P\left(some\ vertex\ has\ degree\ \frac{\log(\mathrm{n})}{\mathrm{loglog}(\mathrm{n})}\right) \geq 1 - (1 - \frac{1}{\mathrm{n}}\mathrm{e}^{-1})^{\mathrm{n}}$$

$$\approx 1 - e^{-\frac{1}{e}} \approx 0.31$$