CS 683 Lecture 29
Date: 4/4/08
Scribes: Seth Marvel, Shaomei Wu

ARROW'S CLAIM AND PROOF

We define a ***global ranking*** (GR) of the set $S = \{a, b, c, \ldots\}$ as a function that maps sets of N orderings on S (e.g. $b > a = c > d > \ldots$) onto single orderings on S. Note that the orderings, or ***voter rankings***, may include both greater than (>) and equality (=) relations.

By definition, a GR must be
  (*i*) total, i.e. for all a and b in S, either $a > b$, $b > a$, or $a = b$ in GR
  (*ii*) transitive, i.e. if $a > b$ and $b > c$ in GR, then $a > c$ in GR

For many applications, we would like the GR to reflect "the general opinion" of the voter rankings. Hence, we may seek GRs that satisfy the following 3 axioms:

(1) <u>Non-Dictator Axiom</u>: None of the voter rankings is a dictator of GR.
For all a and b in S, let an ***ab-dictator*** of GR be a voter ranking where (1) the order of a and b in GR is the same as that in the *ab*-dictator, and (2) no changes to the order of a and b in other voter rankings can change the order of a and b in the GR. A ***dictator*** of GR is an *ab*-dictator of GR for all a and b in S.
(2) <u>Unanimity Axiom</u>: For all a and b in S, if $a > b$ in all voter rankings, then $a > b$ in GR.
(3) <u>Axiom of Independence of Irrelevant Alternatives (AIIA)</u>: For all a and b in S, the relative order of a and b in GR depends only on the relative order of a and b in the voter rankings.

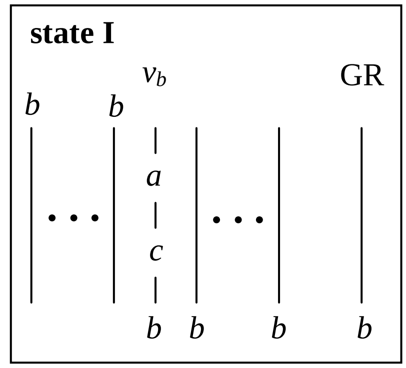
Arrow proved that no GR satisfies all 3 of the above axioms:

<u>Arrow's Theorem</u>: If a GR satisfies (2) and (3), then it violates (1).

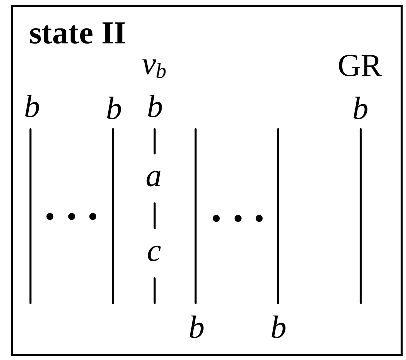
<u>Lemma 1</u>: Consider a set of voter rankings in which each voter ranks b first or last. Then b is either first or last in GR.

<u>Proof</u>: For suppose not. Then there exists some a and c in S such that $a > b > c$ in GR, so $a > c$ in GR (by transitivity of GR). Suppose that we reorder a and c such that $c > a$ in each voter ranking while keeping b in its original position as first or last. Then, by the Unanimity Axiom, $c > a$ in GR. On the other hand, the relative orders of a and b, and b and c are unaffected by re-ranking a and c since b keeps its extreme position. So, by AIIA, we still have $a > b$ and $b > c$ in GR, implying $a > c$ in GR (again, by transitivity). The contradiction proves Lemma 1.

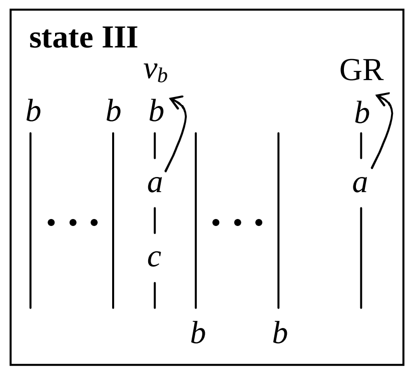
Now consider a set of voter rankings in which each voter assigns b last place. By the Unanimity Axiom, b must also be last in GR. We now select an order for the voters and sequentially move the b's from last to first in each ranking. When all b's have been have been moved to first, the Unanimity Axiom guarantees b is first in the GR. By Lemma 1, then, there exists a voter ranking v_b such that moving b from last to first in v_b *first* moves b from last to first in GR.



Let ***state I*** denote the set of voter rankings just before b is moved from last to first in v_b, and ***state II*** denote the set of voter rankings just after b is moved from last to first in v_b.



Lemma 2: v_b is an *ac*-dictator of GR in state II.



Proof: Without loss of generality, suppose $a > c$ in v_b. We first prove that $a > c$ in GR in state II, the first condition necessary for v_b to be an *ac*-dictator. Suppose we move a above b in v_b so that $a > b$ in v_b, thereby creating ***state III***. In state III, a and b have the same order in all voter rankings as they did in state I, when b was last in GR and $a > b$. So, by AIIA, $a > b$ in GR in state III. Furthermore, in state III, b and c have the same order in all voter rankings as they did in state II, so by AIIA, $b > c$ in GR in state III, implying $a > c$ in GR in state III. Finally, the transition from state II to state III does not change the order of a and c in any voter rankings, so by AIIA, $a > c$ in GR in state II.

To satisfy the second condition for v_b to be an *ac*-dictator, consider 2 cases:

Case 1: Starting with state II, suppose we change the order of a and c in an arbitrary number of voter rankings other than v_b, while leaving b in its extreme position in each of these rankings. Since the relative order of a and b, and b and c, does not change in any ranking by this rearrangement, the above proof of the first condition goes through and $a > c$ in GR regardless of the rearrangement.

Case 2: Suppose we do another rearrangement similar to that in Case 1, but this time do not require that b remain in its extreme position in all voter rankings. If this case were to change $a > c$ to $a = c$ or $a < c$ in GR, then AIIA would be violated since the order of a and c would depend on their order relative to b. Hence, $a > c$ in GR regardless of these rearrangements.

Lemma 2 asserts that in state II, v_b is an *ac*-dictator of GR for all a and c that are not b. If we repeat the proof of Lemma 2 with c playing the role of b, we obtain an *ab*-dictator v_c. As the final step in proving Arrow's Theorem, we prove the following lemma:

Lemma 3: $v_b = v_c$

Proof: Suppose $v_b \neq v_c$ and, without loss of generality, suppose we encounter v_b before v_c in our selected order of voters. Consider a set of voter rankings in which each voter initially ranks *a* in last place. We now sequentially move the *a*'s from last to first in each ranking. If the voter rankings all agree that $c > b$, then by the Unanimity Axiom, $c > b$ in GR. Furthermore, just after *a* is moved from last to first in the *ac*-dictator v_b, v_b ranks $a > c$, so it must be that $a > c$ in GR. By transitivity of GR, this implies $a > b$ in GR. However, v_c is an *ab*-dictator and ranks $b > a$ before *a* is moved from last to first in its ranking, so by contradiction, $v_b = v_c$.

HARE VOTING SYSTEM
(used in faculty rankings at Cornell)

We describe this system via an example. Suppose:
4 individuals rank: $a > b > c$
3 individuals rank: $b > c > a$
2 individuals rank: $c > b > a$
Now, examine the first column. Element *c* has the fewest first-place votes, so we eliminate *c* from all columns. Next, we tally the individuals who rank *b* first among the remaining elements (5) and those who instead rank *a* first (4). Since the former is greater than the latter, *a* is eliminated, leaving *b* the winner.

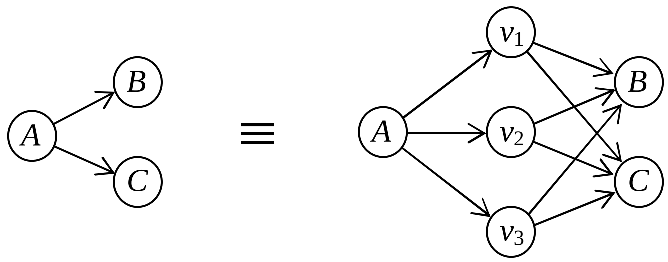
Unfortunately, the Hare Voting System can be sabotaged, and the mechanism of sabotage sometimes takes a bizarre form. Suppose:
7 individuals rank: $a > b > c > d$
6 individuals rank: $b > a > c > d$
5 individuals rank: $c > b > a > d$
3 individuals rank: $d > c > b > a$
By the algorithm described above, *d* is eliminated first, then *b* and finally *c*, leaving *a*. However, if the 3 individuals who voted $d > c > b > a$ would like to prevent *a* from winning and know how the other individuals are voting, they can sabotage *a*'s victory by moving *a*'s rank from last to first: $a > d > c > b$. If the voting process is then repeated, *d* is again eliminated first, then *c*, and finally *a*, to leave *b* the winner.

PAGERANK

Suppose our goal is to design an algorithm α that ranks vertices on a directed, strongly connected, unweighted graph. Altman and Tennenholtz[1] proposed 5 intuitively desirable axioms for α and proved only PageRank satisfies all 5 axioms.

Axiom 1: α assigns equal rank to isomorphic vertices.
Axiom 2: Adding a self-loop to vertex v does not change relative ranking of any other vertices.
Axiom 3: Ideal voting by committee ranks the vertices in the same order as direct voting.

e.g.,



On Monday: Axioms 4 and 5, plus a demonstration that PageRank is satisfied by all 5 axioms.

[1] A. Altman, M. Tennenholtz. Ranking Systems: The PageRank Axioms. In *Proceedings of the 6th ACM Conference on Electronic Commerce* (EC '05).