CS 683: Advanced Design & Analysis of Algorithms February 18, 2008

Lecture 13

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1 Recall on Growth Models

δ is the probability that we create an edge at a given step.

$\forall k \in \mathbb{N}, \forall t \in \mathbb{N}, d_k(t)$ is the expected number of vertices of degree k at time t.

We have :

$$d_0(t+1) = d_0(t) + 1 - 2\delta \frac{d_0(t)}{t}$$
$$d_k(t+1) = d_k(t) + 2\delta \frac{d_{k-1}(t)}{t} - 2\delta \frac{d_k(t)}{t}$$

Then suppose that $d_k = p_k t$,

$$p_0 = \frac{1}{1+2\delta}$$
$$p_k = \left(\frac{2\delta}{1+2\delta}\right)^k p_0 = \frac{1}{1+2\delta}\left(\frac{2\delta}{1+2\delta}\right)^k$$

2 Generating Function for Component Sizes

2.1 Definition

Let $N_k(t)$ the expected number of components of size k.

$$N_1(t+1) = N_1(t) + 1 - 2\delta \frac{N_1(t)}{t}$$
$$N_k(t+1) = N_k(t) + \sum_{j=1}^{k-1} \delta \frac{jN_j(t)}{t} \frac{(k-j)N_{k-j}(t)}{t} - 2\delta \frac{kN_k(t)}{t}$$

Components of size 1 are just vertices of degree 0, thus $N_1 = d_0$. To calculate $N_k(t+1)$ we need to add to $N_k(t)$ all the components of size k created by a new edge, and substract all components of size k where a new edge is added (because this will increase the size of the component).

2.2 Building the Generating Function

Let us suppose that $N_k(t) = a_k t$.

Then

$$a_1(t+1) = a_1 t + 1 - 2\delta a_1$$
$$a_1 = \frac{1}{1+2\delta}$$

And for $k > 1$,

$$a_k(t+1) = a_k t + \sum_{j=1}^{k-1} \delta j a_j (k-j) a_{k-j} - 2\delta k a_k$$
$$a_k = \frac{\delta}{1+2\delta} \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j}$$

$a_k t$ is the expected number of components of size k at time t. $k a_k t$ is the expected number of vertices in a component of size k at time t.

Then $\frac{k a_k t}{t} = k a_k$ is the proportion of vertices in a component of size k at time t. It is consequently the probability that a vertex selected uniformly at random is in a component of size k. This is independant of t.

Then $\sum_{k=1}^{\infty} k a_k = 1$.

We can define the generating function for the size of the component containing a vertex :

$$g(X) = \sum_{k=1}^{\infty} k a_k X^k$$

How to get $g(X)$ in closed form ?

2.3 Derivative of $g(X)$

When evaluating $g(X)$ using the a_k recursion formula, we obtain the following differential equation (see appendix) :

$$g(X) = -2\delta X g'(X) + 2\delta X g(X) g'(X) + X$$

Then we calculate $g'(X)$:

$$g'(X) = \frac{1}{2\delta X} \frac{g(X) - X}{g(X) - 1} = \frac{1}{2\delta} \frac{\frac{g(X)}{X} - 1}{g(X) - 1}$$

From the previous lectures, we know that $g'(1)$ is the expected size of a finite component and $g(1) = 1$ if there is no giant component. We can study g' considering a critical value of δ called δ_{critical} such that if $\delta = \delta_{\text{critical}}$ then $g(1) = 1$.

2.4 Analysis of $g'(1)$

2.4.1 If $\delta > \delta_{\text{critical}}$

Then $g(1) < 1$, then $g'(1) = \frac{1}{2\delta}$ and there is a giant component.

2.4.2 If $g(1) = 1$

$$g'(1) = \frac{1}{2\delta} \left. \frac{\frac{g(X)}{X} - 1}{g(X) - 1} \right|_{x=1}$$

This is an undetermined value, we can use the hospital theorem. This theorem says that the limit of the quotient above is equal to the limit of the quotient of the derivatives at the same point (if this limit exists).

Then,

$$g'(1) = \frac{1}{2\delta} \left. \frac{\frac{Xg'(X) - g(X)}{X^2}}{g'(X)} \right|_{x=1} = \frac{1}{2\delta} \frac{g'(1) - g(1)}{g'(1)}$$

We get the simple equation $g'(X)^2 = \frac{1}{2\delta} g'(1) - \frac{1}{2\delta} g(1)$

The solution is

$$g'(1) = \frac{\frac{1}{2\delta} \pm \sqrt{\frac{1}{4\delta^2} - 4\frac{1}{2\delta}}}{2} = \frac{1 \pm \sqrt{1-8\delta}}{4\delta}$$

This solution is defined if $\delta \leq \delta_{\text{critical}} = \frac{1}{8}$

We also know that $g'(1) = 1$ when $\delta = 0$ (because there are only vertices and no edge), then the solution is (see the asymptotic calculus for the justification) :

$$g'(1) = \frac{1 - \sqrt{1-8\delta}}{4\delta}$$

Then when $\delta = \frac{1}{8}$, $g'(1) = 2$, and we can calculate the equivalent when $\delta \to 0$:

$$\begin{aligned} g'(1) &= \frac{1 - \left[1 - \frac{1}{2}8\delta - \frac{1}{8}64\delta^2 + \dots\right]}{4\delta} \\ &= \frac{4\delta + 8\delta^2 + \dots}{4\delta} \\ g'(1) &\underset{\delta \to 0}{\sim} 1 + 4\delta \end{aligned}$$

3 Molley-Reed model

Let us study Growth Model networks with the probability p_k defined in the first section. The Molley-Reed model stipulates that :

$$\sum_{i=0}^{\infty} i(i-2)p_i = 0$$

Consider p_k for $\delta = \frac{1}{4}$:

$$\forall k \in \mathbb{N}, p_k = \frac{1}{1+2\delta}\left(\frac{2\delta}{1+2\delta}\right)^k = \frac{2}{3}\left(\frac{1}{3}\right)^k$$

Then :

$$\begin{aligned} \sum_{i=0}^{\infty} i(i-2)p_i &= \frac{2}{3}\sum_{i=0}^{\infty} \frac{i(i-2)}{3^i} \\ &= \sum_{i=0}^{\infty} \frac{i^2}{3^i} - 2\sum_{i=0}^{\infty} \frac{i}{3^i} \end{aligned}$$

Consider the different sums (calculated using derivates from the first one) :

$$\begin{aligned} 1 + a + a^2 + \dots &= \frac{1}{1-a} \\ a + 2a^2 + 3a^3 + \dots &= \frac{a}{(1-a)^2} \\ a + 4a^2 + 9a^3 + \dots &= \frac{a(1-a)}{(1-a)^3} \end{aligned}$$

Then we have with $a = \frac{1}{3}$:

$$\sum_{i=0}^{\infty} \frac{i(i-2)}{3^i} = \frac{3}{2} - 2 \times \frac{3}{4} = 0$$

Differential Equation on $g(X)$

We have :

$$\begin{aligned} a_1 &= \frac{1}{1+2\delta} \\ a_k &= \frac{\delta}{1+2k\delta} \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j} \end{aligned}$$

Then let us use it in $g(X)$:

$$\begin{aligned} g(X) &= \sum_{k=1}^{\infty} k a_k X^k \\ &= a_1 X + \sum_{k=2}^{\infty} k \frac{\delta}{1+2k\delta} \sum_{j=1}^{k-1} j(k-j) a_j a_{k-j} X^k \\ &= a_1 X + \sum_{j=1}^{\infty} \sum_{k=j+1}^{\infty} \frac{\delta}{1+2k\delta} kj(k-j) a_j a_{k-j} X^k \\ &= a_1 X + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\delta}{1+2(k+j)\delta} (k+j) jk a_j a_k X^{k+j} \end{aligned}$$

$$\begin{aligned} g(X) &= a_1 X + \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\delta}{1+2(k+j)\delta} k^2 j a_j a_k X^{k+j} \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\delta}{1+2(k+j)\delta} j^2 k a_j a_k X^{k+j} \right) \\ &= a_1 X + \delta \left(\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(1 - \frac{2(k+j)\delta}{1+2(k+j)\delta} \right) k^2 j a_j a_k X^{k+j} \right. \\ &\quad \left. + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(1 - \frac{2(k+j)\delta}{1+2(k+j)\delta} \right) j^2 k a_j a_k X^{k+j} \right) \end{aligned}$$

$$
\begin{aligned}
g(X) = a_1 X + \delta \Bigg(& \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^2 j a_j a_k X^{k+j} + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} j^2 k a_j a_k X^{k+j} \\
& - \Bigg(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{2(k+j)\delta}{1+2(k+j)\delta} k^2 j a_j a_k X^{k+j} \\
& + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{2(k+j)\delta}{1+2(k+j)\delta} k^2 j a_j a_k X^{k+j} \Bigg)\Bigg) \\
= a_1 X + 2\delta & \left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} k^2 j a_j a_k X^{k+j} \right) \\
& - \delta \left(\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{2(k+j)\delta}{1+2(k+j)\delta} (k^2 j + k j^2) a_j a_k X^{k+j} \right)
\end{aligned}
$$

$$
\begin{aligned}
g(X) &= a_1 X + 2\delta \sum_{j=1}^{\infty} j a_j X^j \sum_{k=1}^{\infty} k^2 a_k X^k - \delta \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} \frac{2k\delta}{1+2k\delta}(k-j) j k a_j a_{k-j} X^k \\
&= a_1 X + 2\delta X g(X) g'(X) - \delta \sum_{k=2}^{\infty}\sum_{j=1}^{k-1} \frac{2k\delta}{1+2k\delta}(k-j) j k a_j a_{k-j} X^k \\
&= a_1 X + 2\delta X g(X) g'(X) - \sum_{k=2}^{\infty} 2k^2 \delta a_k X^k \\
&= a_1 X + 2\delta X g(X) g'(X) - 2\delta X (g'(X) - a_1) \\
&= a_1(1+2\delta) X + 2\delta X g(X) g'(X) - 2\delta X g'(X) \\
&= X + 2\delta X g(X) g'(X) - 2\delta X g'(X)
\end{aligned}
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(This calculus has been done by Jean-Baptiste Jeannin, Maxime Brugidou and Pierre-Alexandre Meyer).