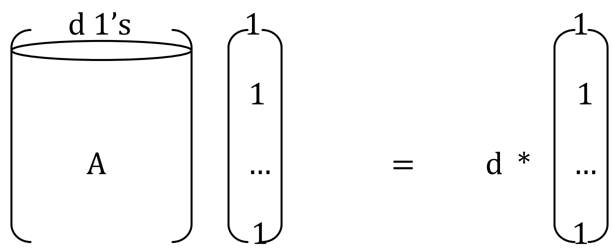
CS683 3/31/08 Monday Scribe: Haibo Lu

Lemma Let G be a regular degree d connected undirected graph with adjacency matrix A. The eigenvalues of A satisfy:

$$d = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n \geq -d$$

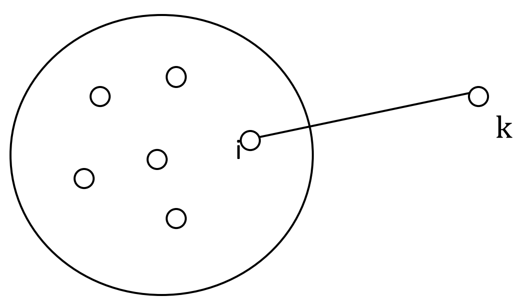
where, λ_n = -d iff G is bipartite.

Proof Let u = (1,1,…,1). u is an eigenvector of A with eigenvalue d.

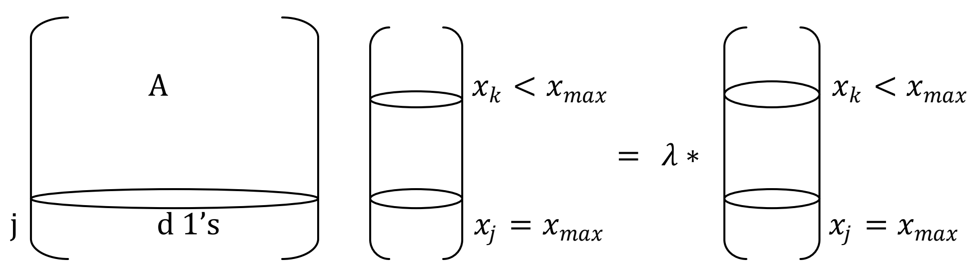


Let x be an eigenvector not proportional to u. Let x_{max} be maximum coordinate of x:

$$S = \{i | x_i = x_{max}\}$$



$$Ax = \lambda x$$



$$\lambda x_j < dx_{max}$$

$$\lambda x_{max} = dx_{max}$$

$$\lambda < d$$

Lemma Let G be a regular degree d undirected graph with adjacency matrix A and k components. Then $d = \lambda_1 = \lambda_2 = \lambda_k > \lambda_{k+1} \ldots$

$$\begin{pmatrix} B1 & 0 & 0 \\ 0 & B2 & 0 \\ 0 & 0 & B3 \end{pmatrix} \begin{pmatrix} x1 \\ x2 \\ x3 \\ 0 \\ \ldots \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda x1 \\ \lambda x2 \\ \lambda x3 \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

$$B_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Adding edges to a graph can only increase the largest eigenvalue.

Lemma Let G_1 and G_2 be graphs where $G_1 \subseteq G_2$. The maximum eigenvalues of G_2 is at least as large as maximum eigenvalue of G_1.

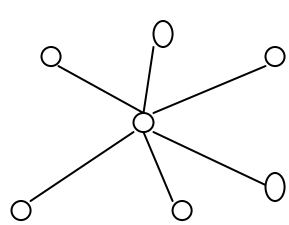
Proof Let A_1 and A_2 be adjacency matrix of G_1 and G_2. Let $\lambda_1(A_1)$ and $\lambda_1(A_2)$ be largest eigenvalues. Let v be eigenvector associated with $\lambda_1(A_1)$. We can show v has all non-negative coordinates.

Since v has all non-negative coordinates,

$\lambda_1(A_1) = v^T A_1 v \le v^T A_2 v$ (because A_2 has more 1's than A_1)

But $\lambda_1(A_2) = \max_{|x|=1} x^T A_2 x \ge v^T A_2 v \ge v^T A_1 v = \lambda_1(A_1)$.

Star



$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 1 & & 0 & & \\ \dots & & & & \\ 1 & & & & \end{pmatrix}\begin{pmatrix} \sqrt{n-1} \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} = \begin{pmatrix} n-1 \\ \sqrt{n-1} \\ \sqrt{n-1} \\ \dots \\ \sqrt{n-1} \end{pmatrix} = \sqrt{n-1}\begin{pmatrix} \sqrt{n-1} \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & & & & \\ 1 & & 0 & & \\ \dots & & & & \\ 1 & & & & \end{pmatrix}\begin{pmatrix} -\sqrt{n-1} \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} = \begin{pmatrix} n-1 \\ -\sqrt{n-1} \\ -\sqrt{n-1} \\ \dots \\ -\sqrt{n-1} \end{pmatrix} = -\sqrt{n-1}\begin{pmatrix} -\sqrt{n-1} \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix}$$

Eigenvalues of Star: $\sqrt{n-1}, 0, 0, \dots, 0, -\sqrt{n-1}$

__Theorem__ $\max \{d_{min}, \sqrt{d_{max}}\} \le \lambda_1 \le \min \{d_{max}, \sqrt{2|E|}\}$

__Proof__ Let $u = (1,1,\ldots,1)$

$$Au \ge d_{min}\, u \quad , \quad u^T A u \ge d_{min}\, u^T u$$

$$\lambda_1 = \max_x \frac{x^T A x}{x^T x} \ge \frac{u^T A u}{u^T u} \ge d_{min}$$

Let G_s be star consisting of highest degree vertex of degree d_{max}. The maximum eigenvalue of A_s is $\sqrt{d_{max}}$. Since $G_s \subseteq G$, maximum eigenvalue is at least $\sqrt{d_{max}}$.

Now, let's prove the upper bound:

Let v_1 be the first eigenvector normalized so that maximum coordinate is 1.

$$u = (1,1,\dots,1), \quad \lambda v_1 = A v_1 \le Au \le d_{max}\, u$$

$$\lambda \le d_{max}$$

Meanwhile, $\lambda_1 = \max_{|x|=1} x^T A x = |A|_2 \le |A|_F = \sqrt{\sum a_{ij}^2} = \sqrt{2|E|}$.