CS 683: Advanced Design & Analysis of Algorithms 2/20/2008

Lecture 14

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1 Condition for phase transition to appear

Given

$$h_1 = xg_1(h_1)$$

we can derivate this expression and we have

$$h_1'(x) = g_1(h_1(x)) + xg_1'(h_1(x))\, h_1'(x) \tag{1}$$

Given that $h_1(1) = 1$ and $g_1(1)$=1, we can replace them in equation 1 and we have

$$h_1'(1) = 1 + g_1'(1)h_1'(1) \tag{2}$$

If we derivate the expression of $h_0(x)$, we have

$$h_0'(x) = g_0(h_1(x)) + xg_0'(h_1(x))\, h_1'(x) \tag{3}$$

and

$$h_0'(1) = 1 + g_0'(1)h_1'(1) \tag{4}$$

Thanks to equations 2 and 4, we have finally

$$h_1'(1) = \frac{1}{1 - g_1'(1)} \tag{5}$$

and

$$h_0'(1) = 1 + \frac{g_0'(1)}{1 - g_1'(1)} \tag{6}$$

Given that $h_0'(1)$ is the average size of a component given a vertex chosen uniformly at random, we have proved that the phase transition occurs when the formulas *explodes*, that is when $g_1'(1) = 1$.

2 Apparition of the phase transition

Given that

$$g_1(x) = \frac{g_0'(x)}{g_0'(1)} \tag{7}$$

the following condition must hold for the phase transition to occur

$$g_0'(1) = g_0''(1) \tag{8}$$

But, by definition, we have

$$g_0(x) = \Sigma_k p_k x^k \tag{9}$$

$$g_0'(x) = \Sigma_k k p_k x^{k-1} \tag{10}$$

$$g_0''(x) = \Sigma_k k(k-1) p_k x^{k-2} \tag{11}$$

Finally, condition 8 can be re-written as

$$\Sigma_k k(k-2)p_k = 0$$

which is the Molley-Reed condition.

2.1 Special case: G(n,p)

In the special case of a random graph $G(n, p)$, with $d = np$, we have

$$p_k = \exp^{-d} \frac{d^k}{k!}$$

$g_0'(1)$ is the expected degree which is d in our case.
Equation 7 gives us

$$g_1(x) = \frac{1}{d} \Sigma_k k \exp^{-d} \frac{d^k}{k!} x^{k-1} = g_0(x)$$