1 Condition for phase transition to appear

Given

\[ h_1 = xg_1 (h_1) \]

we can derivate this expression and we have

\[ h'_1(x) = g_1 (h_1(x)) + xg'_1 (h_1(x)) h'_1(x) \]  \hspace{1cm} (1)

Given that \( h_1(1) = 1 \) and \( g_1(1)=1 \), we can replace them in equation 1 and we have

\[ h'_1(1) = 1 + g'_1(1)h'_1(1) \]  \hspace{1cm} (2)

If we derivate the expression of \( h_0(x) \), we have

\[ h'_0(x) = g_0 (h_1(x)) + xg_0 (h_1(x)) h'_1(x) \]  \hspace{1cm} (3)

and

\[ h'_0(1) = 1 + g'_0(1)h'_1(1) \]  \hspace{1cm} (4)

Thanks to equations 2 and 4, we have finally

\[ h'_1(1) = \frac{1}{1 - g'_1(1)} \]  \hspace{1cm} (5)

and

\[ h'_0(1) = 1 + \frac{g'_0(1)}{1 - g'_1(1)} \]  \hspace{1cm} (6)

Given that \( h'_0(1) \) is the average size of a component given a vertex chosen uniformly at random, we have proved that the phase transition occurs when the formulas \textit{explode}, that is when \( g'_1(1) = 1 \).

2 Apparition of the phase transition

Given that

\[ g_1(x) = \frac{g'_0(x)}{g'_0(1)} \]  \hspace{1cm} (7)

the following condition must hold for the phase transition to occur

\[ g'_0(1) = g''_0(1) \]  \hspace{1cm} (8)

But, by definition, we have

\[ g_0(x) = \sum_k p_k x^k \]  \hspace{1cm} (9)

\[ g'_0(x) = \sum_k k p_k x^{k-1} \]  \hspace{1cm} (10)

\[ g''_0(x) = \sum_k k(k-1) p_k x^{k-2} \]  \hspace{1cm} (11)

Finally, condition 8 can be re-written as

\[ \sum_k k(k-2) p_k = 0 \]

which is the Molley-Reed condition.
2.1 Special case: $G(n,p)$

In the special case of a random graph $G(n,p)$, with $d = np$, we have

$$p_k = \exp^{-d} \frac{d^k}{k!}$$

$g'_0(1)$ is the expected degree which is $d$ in our case.

Equation 7 gives us

$$g_1(x) = \frac{1}{d} \sum_k k \exp^{-d} \frac{d^k}{k!} x^{k-1} = g_0(x)$$