2/20/2008 CS683 Part I Haibo Lu

Grown graph

$N_k(t)$ = number of components of size k
Let's write $N_k(t)$ as $N_k(t) = a_k t$, t is time/steps elapsed

Generating function for $N_k(t)$

$$g(x) = \sum_{k=0}^{\infty} k a_k x^k$$

Use g(x) to solve for a_k as recurrence equation:

$$g = -2\delta x g' + 2\delta x g g' + x$$

$$g' = \frac{1-\frac{g}{x}}{1-g} * \frac{1}{2\delta}$$

$g'(1)$ = expected size of finite components

(1) $\delta > \delta_{critical}$: Giant component apppears

$$g(1) = 1 \ , \qquad g'(1) = \frac{1}{2\delta}$$

(2) $\delta \le \delta_{critical}$: Only finite components

$$g'(1) = \lim_{x\to 1} \frac{1}{2\delta} \frac{1-\frac{g(x)}{x}}{1-g(x)} = \frac{1}{2\delta} \lim_{x\to 1} - \frac{\frac{g'x-g}{x^2}}{-g'} \text{ (use L^Hospital Law)}$$

$$= \frac{1}{2\delta} lim_{x\to 1} \frac{g(1)-g'(1)}{-g'(1)}$$

$$= \frac{1}{2\delta} * \frac{g'(1)-g(1)}{g'(1)}$$

$$[g'(1)]^2 - \frac{1}{2\delta} g'(1) + \frac{1}{2\delta} g(1) = 0$$

$$g'(1) = \frac{1 \mp \sqrt{1-8\delta}}{4\delta} \text{ (only retain '-' term)}$$

We can see $g'(1)$ only has real solution if $\delta \le \frac{1}{8}$. $\delta_{critical} = \frac{1}{8}$.

Derivation of Molley-Reed condition

Consider four generating functions:

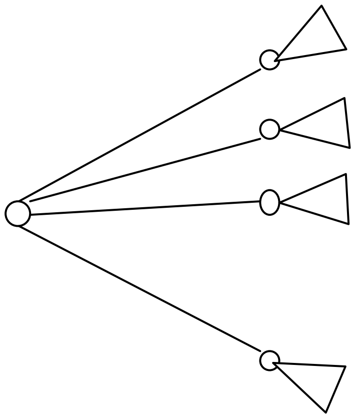
- g_0 = degree of vertex chosen uniformly at random
- g_1 = out degree of vertex at end of random edge
- h_0 = size of component containing vertex chosen uniformly at random
- h_1 = size of component at end of edge

$$g_0 = \sum_{k=0}^{\infty} p_k x^k,$$

where p_k is probability that a vertex chosen a random is of degree k

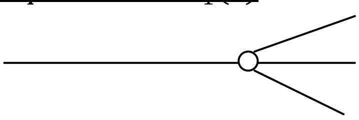
$$g_1 = \frac{1}{x}\frac{\sum_{k=1}^{\infty} k p_k x^k}{\sum_{k=1}^{\infty} k p_k} = \frac{\sum_{k=0}^{\infty} k p_k x^{k-1}}{\sum_{k=0}^{\infty} k p_k} = \frac{g_0'(x)}{g_0'(1)} \qquad \text{(A)}$$

Probability distribution for number of vertices at distance 2 from randomly chosen vertices:



$$\sum_{k=0}^{\infty} p_k (g_1(x))^k = g_0(g_1(x))$$

Equation for $h_1(x)$

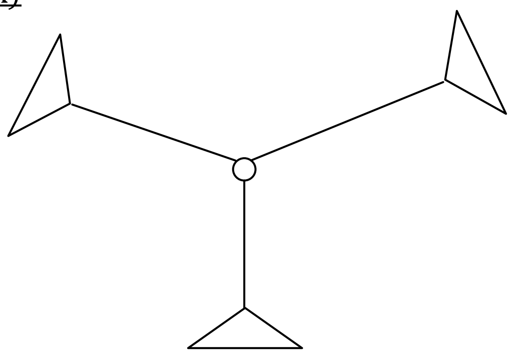


Let q_k be probability of k outgoing edges

$$x[q_0 + q_1 h_1(x) + q_2 \big(h_1(x)\big)^2 + \cdots]$$

$$h_1(x) = x g_1(h_1(x))$$

Equation for $h_0(x)$



$$h_0(x) = x\sum_{k=0}^{\infty} p_k (h_1(x))^k = xg_0(h_1(x))$$

(Continue on Part II)