In the spring of 1930,. . . König told me that he was about to finish a book that would include all that was known about graphs. I assured him that such a book would fill a great need; and I brought up my n-Arc Theorem which, having been published as a lemma in a curve-theoretical paper, had not yet come to his attention. König was greatly interested, but did not believe the theorem was correct. "This evening," he said to me in parting, "I won't go to sleep before having constructed a counterexample!". When me met again the next day he greeted me with the words "Sleepless night!"

— Karl Menger, "On the origin of the n-arc theorem", *J. Graph Theory* 5:341–350, 1981.

10 Graph Separators

"Divide and conquer" is one of the oldest and most widely used techniques for designing efficient algorithms. Divide-and-conquer algorithms partition their inputs into two or more independent subproblems, solve those subproblems recursively, and then combine the solutions to those subproblems to obtain their final output. This strategy can be successfully applied to several graph problems, provided we can quickly *separate* the graph into roughly equal subgraphs.

An ***ε*-separator** of an n-vertex graph $G = (V, E)$ is a subset $S \subseteq V$ such that each connected component of $G \setminus S$ has at most εn vertices. Our goal is to find ε-separators, for some constant $1/2 \leq \varepsilon < 1$, that have few vertices. For example, any path has a $1/2$-separator consisting of a single vertex; any binary tree has a $2/3$-separator consisting of a single vertex; and any outer-planar graph has a $2/3$-separator consisting of two vertices.

The following classical theorem of Menger [12], which is both a precursor and an easy consequence of the maxflow-mincut theorem, is a key tool in proving the existence of small separators.

Theorem 10.1 (Menger). *Let $G = (V, E)$ be a graph. The minimum number of vertices separating any subsets $A, b \subseteq V$ is equal to the maximum number of vertex-disjoint paths from A to B.*

10.1 Planar Separators

In the late 1970s, Richard Lipton and Robert Tarjan [11] proved the following seminal result.

The Planar Separator Theorem. *Any n-vertex planar graph has a $2/3$-separator containing at most $\sqrt{8n}$ vertices.*

Proof (Alon, Seymour, and Thomas [2]): Let G be an embedded planar graph with $n \geq 3$ vertices, and let $k = \lfloor \sqrt{2n} \rfloor$. Without loss of generality, we can assume that G has no loops or parallel edges, and that every face is a triangle bounded by three distinct edges. For any simple cycle C in G, let $In(C)$ and $Out(C)$ denote the vertices inside and outside C, respectively. No vertex of $In(C)$ is adjacent to any vertex of $Out(C)$. Let C be a simple cycle satisfying three conditions:

(1) C has at most $2k$ vertices.

(2) $|Out(C)| < 2n/3$.

(3) Subject to conditions (1) and (2), the difference $|In(C)| - |Out(C)|$ is as small as possible.

The outer face of G satisfies the conditions (1) and (2), so an appropriate cycle C exists.

For purposes of deriving a contradiction, suppose $|In(C)| \geq 2n/3$. Let D be the subgraph of G in the closed interior of C. For any two vertices u and v in C, let $c(u, v)$ denote their distance in C, and let $d(u, v)$ denote their distance in d. The remainder of the proof rests on two claims about the cycle C.

Claim 1. *$d(u,v) = c(u,v)$ for all vertices u and v in C.*

We clearly have $d(u,v) \le c(u,v)$ for all u and v, because C is a subgraph of D. Suppose there are distinct vertices u and v such that $d(u,v) < c(u,v)$; choose such a pair so that $d(u,v)$ is minimized. Let σ be a shortest path from u to v in D. If σ contained any other vertex w in C, then

$$d(u,w) + d(w,v) \;=\; d(u,v) \;<\; c(u,v) \;\le\; c(u,w) + c(w,v),$$

so either $d(u,w) < c(u,w)$ or $d(w,v) < c(w,v)$; either possibility contradicts our choice of u and v. Thus, σ cannot contain and other vertices of C. it follows that σ cuts D into two smaller disks; call their bounding cycles C^+ and C^-. Suppose $|In(C^+)| \ge |In(C^-)|$. The cycle C^+ has fewer vertices than C, because $d(u,v) < c(u,v)$, so C^+ satisfies condition (1). We also have

$$\begin{aligned} n - |Out(C^+)| &= |In(C^+)| + |V(C^+)| \\ &> \frac{1}{2}\Big(|In(C^+)| + |In(C^-)| + |V(\sigma)| - 2\Big) \\ &= \frac{|In(C)|}{2} \ge \frac{n}{3}, \end{aligned}$$

so C^+ also satisfies condition (2). Finally, we have $|In(C^+)| < |In(C)|$ and $|Out(C^+)| > |Out(C)|$. But this contradicts condition (3) of C. We conclude that Claim 1 is true.

Claim 2. *C has exactly $2k$ vertices.*

Suppose to the contrary that C has strictly less than $2k$ vertices. Choose an arbitrary edge uw of C, and let v be the third vertex of the face of D adjacent to uw. Let α denote the path $uv \cdot vw$, and let β denote the path $C \setminus uw$. These two paths must be distinct, because $In(C) \ne \varnothing$; thus, by Claim 1, v is not a vertex of C. It follows that $C' = \alpha \cdot \beta$ is a simple cycle satisfying conditions (1) and (2). But we also have $|In(C')| < |In(C)|$ and $|Out(C')| = |Out(C)|$, contradicting condition (3) of C. We conclude that Claim 2 is true.

We now return to the main proof. Let $v_0, v_1, \ldots, v_{2k-1}, v_{2k} = v_0$ be the vertices of C in order. Claim 1 implies that the shortest path in D from v_0 to v_k has length k. Thus, by Menger's Theorem, there are $k+1$ vertex-disjoint paths in D from $\{v_0, v_1, \ldots, v_k\}$ to $\{v_k, v_{k+1}, \ldots, v_{2k}\}$. Call these paths $\pi_0, \pi_1, \ldots, \pi_k$, where π_i is a path from v_i to v_{2k-i}. Claim 1 now implies that

$$n \;\ge\; \sum_{i=0}^{k} \big|V(\pi_i)\big| \;\ge\; \sum_{i=0}^{k} \min\{2i+1,\; 2k-2i-1\} \;\ge\; \frac{(k+1)^2}{2},$$

which implies that $k \le \sqrt{2n} - 1$. But this contradicts our definition $k = \lfloor \sqrt{2n} \rfloor$.

We conclude that our assumption $|In(C)| \ge 2n/3$ must be false. Thus, C is a $2/3$-separator of G of size $2\lfloor \sqrt{2n} \rfloor$. □

10.2 Planar Separators, Take 2

A particularly elegant proof of the planar separator theorem, with slightly weaker constants, was discovered by Dan Spielman and Shang-Hua Teng in 1996 [16]. Their proof relies on a geometric characterization of planar graphs, first published by Koebe in 1936 [10], and later independently rediscovered by Andreev [3, 4] and Thurston [17].

Let $\hat{\mathbb{C}}$ denote the extended complex plane $\mathbb{C} \cup \{\infty\}$; this space is homeomorphic to the sphere S^2 by stereographic projection. A ***Möbius transformation*** is a function $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form $\phi(z) = (a + bz)/(c + dz)$ for some complex numbers a, b, c, d where $ad - bc \neq 0$. Möbius transformations map circles to circles, either on the sphere or in the plane (where a line in the plane is considered a circle through ∞).

Theorem 10.2 (Koebe-Andreev-Thurston). *A graph is planar if and only if it is the intersection graph of a finite set of interior-disjoint circular caps on the sphere. Moreover, this representation is unique up to Möbius transformations from the sphere to itself.*

Theorem 10.3. *Every n-vertex planar graph G has a $3/4$-separator of size at most $2\sqrt{n}$.*

Proof (Speilman and Teng): Let G be an n-vertex planar graph. By the Koebe-Andreev-Thurston theorem, there is a set of circular caps $D_1, D_2, \ldots, D_n$ on the unit sphere S^2, whose intersection graph is G. Let $P = \{p_1, \ldots, p_n\}$ be the set of centers of disks D_i. A theorem of Rado [15] implies that the set P has a a ***center point*** c: Every plane through c partitions P into subsets of size at most $\lfloor 3n/4 \rfloor$. (The bound $\lfloor 3n/4 \rfloor$ is best possible; consider the vertices of a regular tetrahedron.) Miller *et al.* [13] proved that there is a Möbius transformation $\Pi : S^2 \to S^2$ such that the origin is a center point of the set $Q = \Pi(P)$. Let $C_i = \Pi(D_i)$ for all i. Any plane through the origin intersects a subset of the caps C_i; the corresponding vertices of G define a 3/4-separator.

It remains only to show that there is a plane through the origin that intersects at most $2\sqrt{n}$ caps C_i. In fact, we will show that the *expected* number of caps intersecting a *random* plane through the origin is at most $2\sqrt{n}$. In fact, this probabilistic bound holds for *any* collection of n circular caps, even with intersecting interiors; we require only that each cap is no bigger than a hemisphere.

Let r_i denote the radius of the boundary circle of C_i. We have $r_i \leq 1$ for all i, because the origin is a center point for Q. For each cap C_i, let B_i denote the set of unit normal vectors of planes through the origin that intersect C_i. The set B_i is a ***belt*** of points in S^2 between two parallel planes symmetric about the origin; the distance between these planes is exactly $2r_i$. By construction, a plane intersects C_i if and only if its unit normal vector lies in B_i. Thus, we need to show that a random point in S^2 lies in at most $2\sqrt{n}$ belts B_i, on average.

The surface area of belt B_i is exactly $4\pi r_i$, and the surface area of the entire sphere S^2 is 4π. Thus, the probability that a random point in S^2 lies inside B_i is exactly r_i. It follows that the expected number of belts B_i containing a random point in S^2, and thus the expected number of caps C_i intersecting a random plane through the origin, is exactly $\sum_i r_i$. On the other hand, the area of cap C_i is at least πr_i^2. Because the caps C_i have disjoint interiors, we have $\sum_i r_i^2 \leq 4$. Given this constraint, the sum $\sum_i r_i$ is maximized when $r_i = 2/\sqrt{n}$. Thus, the the expected number of caps C_i intersecting a random plane through the origin is at most $2\sqrt{n}$, as claimed. □

10.3 Separators for Surface Graphs

Gilbert, Hutchinson, and Tarjan [8] observed that the following result of Albertson and Hutchinson [1] immediately implies a separator theorem for graphs of small genus. A cycle γ in a surface Σ is ***separating*** if $\Sigma \setminus \gamma$ is disconnected, and *nonseparating* otherwise.

Lemma 10.4. *Any n-vertex triangulation of an orientable surface of positive genus contains a nonseparating cycle of length at most $2\sqrt{n}$.*

Proof: Let σ be the shortest nonseparating cycle in G. The graph $G \setminus\!\!\setminus \sigma$ has two copies of σ; call these cycles $\sigma^\flat$ and $\sigma^\sharp$. Let m denote the number of vertices in σ.

Let S be the smallest set of vertices of $G \setminus\!\!\setminus \sigma$ that separates $\sigma^\flat$ and $\sigma^\sharp$. The induced subgraph $G[S]$ must contain (and therefore must *be*) a cycle τ in G that is homologous with σ and therefore nonseparating. Thus, τ has at least m vertices. Menger's Theorem now immediately implies that there are at least m vertex-disjoint paths from $\sigma^\flat$ to $\sigma^\sharp$ in $G \setminus\!\!\setminus \sigma$.

On the other hand, let π be the shortest path in $G \setminus\!\!\setminus \sigma$ from a node in $\sigma^\flat$ to its clone in $\sigma^\sharp$. The edges of π comprise a nonseparating cycle in G, which implies that π has length at most m. At most $m/2$ edges in π lie in $\sigma^\flat$ or $\sigma^\sharp$. Thus, every path from $\sigma^\flat$ to $\sigma^\sharp$ has at least $m/2$ edges.

We conclude that $n \geq m^2/2$. □

Corollary 10.5. *Any n-vertex graph G embedded on an orientable surface of genus g has a $2/3$-separator S of size at most $2g\sqrt{n} + \sqrt{8n}$. Moreover, every component of $G \setminus S$ is planar.*

Proof: Let G be a graph on an orientable surface of genus g. Without loss of generality, we can assume that the embedding is a cellular triangulation; otherwise, replace any non-disk faces with disks, remove all loops and parallel edges, and triangulate any faces with more than three sides. Let σ be the shortest noncontractible cycle in G, and let G' be the induced subgraph of G obtained by removing the vertices of σ. This graph can be embedded on a surface of genus $g-1$. The inductive hypothesis implies that G' has a $2/3$-separator S' of size at most $2(g-1)\sqrt{n} + \sqrt{8n}$. Thus, $S = S' \cup V(\sigma)$ is a $2/3$-separator of size at most $2g\sqrt{n} + \sqrt{8n}$, as required. The base case for the recursion is the Lipton-Tarjan theorem. □

Hutchinson [9] proved that any surface graph has a noncontractible cycle of length $O(\sqrt{n/g}\log g)$. An easy extension of the proof of Corollary 10.5 implies that any surface graph has a $2/3$-separator of size $O(\sqrt{gn}\log g)$. Przytycka and Przytycki [14] proved that there are surface graphs in which the shortest noncontractible cycle has length $\Omega(\sqrt{n\log g/g})$, so the best bound on separator size one can hope to prove using this technique is $O(\sqrt{gn\log g})$. (As far as I know, a tight bound on the length of the shortest noncontractible cycle is still unknown.)

10.4 Greedy Tree-Cotree Decomposition

Djidjev [5] and Gilbert, Hutchinson, and Tarjan [8] independently proved that any graph embedded on an orientable surface of genus $g > 0$ has a $2/3$-separator of size $O(\sqrt{gn})$, which is optimal up to constant factors. Removing the logarithmic factor from Hutchinson's bound requires considering structures larger than individual cycles. For any graph $G = (V, E)$ embedded on a (possibly nonorientable) surface Σ of Euler genus $\bar{g}$, we define the following ***greedy tree-cotree decomposition***.

Let T be a breadth-first search tree rooted at an arbitrary vertex r. For each vertex v, let $\boldsymbol{P(v)}$ denote the path in T from v to r, and let $\boldsymbol{d(v)}$ denote the length of this path. For any edge uv in $G \setminus T$, let $\ell(uv) = d(u) + d(v) + 1$. We inductively define sequences of edges $\boldsymbol{e_1, e_2, \ldots, e_{\bar{g}}}$ and subgraphs $\boldsymbol{Q_0, Q_1, \ldots, Q_{\bar{g}}}$ as follows. Let $Q_0 = \varnothing$. For any index $i > 0$, let $e_i = u_i v_i$ be an edge of minimum weight $\ell(e_i)$ such that $\Sigma \setminus (Q_{i-1} \cup P(u_i) \cup P(v_i) \cup \{e_i\})$ is connected, and let $Q_i = Q_{i-1} \cup P(u_i) \cup P(v_i) \cup \{e_i\}$. Finally, let $L = \{e_1, e_2, \ldots, e_{\bar{g}}\}$ and $C = G \setminus (T \cup L)$.

Alternately, we can define L and C directly in terms of the graph G in the following way. Let $H_0 = (G \setminus T)^*$. For each index $i > 0$, let e_i be an edge of minimum weight $\ell(e_i)$ such that $H_{i-1} \setminus e_i^*$ is connected, and let $H_i = H_{i-1} \setminus e_i^*$. Finally, let $L = \{e_1, e_2, \ldots, e_{\bar{g}}\}$ and $C^* = H_{\bar{g}}$. It should be clear from this inductive definition that C^* is a *maximum* spanning tree of $(G \setminus T)^*$, where the weight of any dual edge e^* is $\ell(e)$.

It is not hard to prove that these two definitions are equivalent [7]. In particular, for any integer i, the subgraph H_i is a ***retract*** of the surface $\Sigma \setminus Q_i$. Thus, for any edge uv, the graph $H_i \setminus (uv)^*$ is connected if and only if the surface $\Sigma \setminus (Q_i \cup P(u) \cup P(v) \cup \{uv\})$ is connected.

Finally, Euler's formula implies the subgraph $Q := Q_{\bar{g}}$ is a cut graph for Σ. In fact, Q is *almost* a reduced cut graph; the root vertex r might have degree 1 in Q.

10.5 Surface Slicing

Now without loss of generality, assume that G is a triangulation. Define the depth of any edge or face to be the maximum depth of its vertices. For any integers i and j, let $\boldsymbol{G[i,j]}$ be the subgraph of vertices and edges whose whose depth lies in the interval $[i, j]$, and let $\boldsymbol{\Sigma[i,j]}$ denote the surface composed of vertices, edges, and faces whose depth lies in the interval $[i, j]$.

For any integer $1 \le i \le \bar{g}$, define $\boldsymbol{d_i} := d(e_i)$, where e_i is the ith edge in L, and define $d_0 = 0$ and $d_{\bar{g}+1} = \infty$. For any integers i and j, let $\boldsymbol{Q[i,j]} := G[i, j] \cap Q_k$, where $d_k \le j < d_{k+1}$.

Lemma 10.6. *For any integers $i \le j$, the subgraph $G[i, j] \setminus Q[i, j]$ is planar.*

Proof: Any subgraph of a planar graph is planar, so it suffices to consider the special case $i = 0$. Fix an integer j, and let k be the integer such that $d_k \le j < d_{k+1}$ and therefore $Q[0, j] = Q_k$. We will actually prove that the surface $\Sigma[0, j] \setminus\!\!\setminus Q_k$ has genus 0.

A loop in a surface with boundary is ***essential*** if it is not homotopic to a separating cycle; in particular, boundary cycles are inessential. Every surface with positive genus has at least one essential loop.

Suppose $\Sigma[0, j] \setminus\!\!\setminus Q_k$ has positive genus. Let ℓ be the shortest essential loop in $\Sigma[0, j] \setminus\!\!\setminus Q_k$. Exactly one edge e in ℓ does not lie in the breadth-first spanning tree $T \cap G[0, j]$. Moreover, e is an edge of minimum depth that defines an loop that is essential in $\Sigma[0, j] \setminus\!\!\setminus Q_k$, and therefore is also essential in $\Sigma \setminus\!\!\setminus Q_k$. It follows that $e = e_{k+1}$; but this is impossible, because $d(e) \le j < d_{k+1}$. $\square$

For any integers $0 \le i < k$, let $\boldsymbol{D(i,k)}$ denote the set of vertices whose depth mod k is equal to i:

$$D(i, k) := \{v \in V \mid d(v) \bmod k = i\}.$$

Removing the vertices in $D(i, k)$ breaks G into several slices:

$$G \setminus D(i, k) = \bigcup_a G[i + ak + 1,\ i + (a + 1)k - 1].$$

Let $\boldsymbol{Q(i,k)}$ denote the corresponding subgraph of the cut graph Q:

$$Q(i, k) := \bigcup_a Q[i + ak + 1,\ i + (a + 1)k - 1].$$

Finally, let $\boldsymbol{S(i,k)}$ denote the vertex set $D(i, k) \cup V(Q(i, k))$. Lemma 10.6 immediately implies that the subgraph $G \setminus S(i, k)$ is planar, for all i and k.

Lemma 10.7. *For some integers i and k, the set $S(i, k)$ contains at most $2\sqrt{n\bar{g}}$ vertices.*

Proof: Let v be an endpoint of an edge in L. The cut graph $Q(i, k)$ contains the path from v to its nearest ancestor in T that lies in $D(i, k)$; this path has length $(d(v) - i) \bmod k$. Moreover, $Q(i, k)$ is the union of all $2\bar{g}$ such paths. Thus, for any integer k, we have

$$\sum_{i=0}^{k-1} |V(G(i, k))| \;\le\; \sum_{v \in V(L)} \sum_{i=0}^{k-1} (d(v) - i) \bmod k \;=\; \bar{g}k(k - 1) \;<\; \bar{g}k^2.$$

Each vertex of G belongs to exactly one subset $D(i,k)$, so

$$\sum_{i=0}^{k-1} D(i,k) = n.$$

We conclude that

$$\sum_{i=0}^{k-1} |S(i,k)| \le n + \bar{g}k^2,$$

which implies that $|S(i,k)| \le n/k + \bar{g}k$ for some i. In particular, if we set $k = \sqrt{n/\bar{g}}$, then we have $\min_i |S(i,k)| \le 2\sqrt{n\bar{g}}$. □

Corollary 10.8 (Eppstein [6]). *Any n-vertex graph G embedded on a surface of Euler genus $\bar{g}$ has a $2/3$-separator S of size at most $(2\sqrt{\bar{g}} + \sqrt{8})\sqrt{n}$. Moreover, each component of $G \setminus S$ is planar.*

References

[1] M. O. Albertson and J. P. Hutchinson. On the independence ratio of a graph. *J. Graph Theory* 2(1):1–8, 1978.

[2] N. Alon, P. Seymour, and R. Thomas. Planar separators. *SIAM J. Discrete Math.* 7(4):184–193, 1994.

[3] E. M. Andreev. Convex polyhedra in Lobačevskiĭ space. *Math. USSR Sbornik* 10:413–440, 1970.

[4] E. M. Andreev. On convex polyhedra of finite volume in Lobačevskiĭ space. *Math. USSR Sbornik* 12(2):270–259, 1970.

[5] H. N. Djidjev. A separator theorem. *Compt. Rend. Acad. Bulg. Sci.* 34:643–645, 1981.

[6] D. Eppstein. Dynamic generators of topologically embedded graphs. *Proc. 14th Ann. ACM-SIAM Symp. Discrete Algorithms*, 599–608, 2003.

[7] J. Erickson and K. Whittlesey. Greedy optimal homotopy and homology generators. *Proc. 16th Ann. ACM-SIAM Symp. Discrete Algorithms*, 1038–1046, 2005.

[8] J. R. Gilbert, J. P. Hutchinson, and R. E. Tarjan. A separator theorem for graphs of bounded genus. *J. Algorithms* 5(3):391–407, 1984.

[9] J. P. Hutchinson. On short noncontractible cycles in embedded graphs. *SIAM J. Discrete Math.* 1(2):185–192, 1988.

[10] P. Koebe. Kontaktprobleme der Konformen Abbildung. *Ber. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl.* 88:141–164, 1936.

[11] R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM J. Applied Math.* 36(2):177–189, 1979.

[12] K. Menger. Zur allgemeinen Kurventheorie. *Fund. Math.* 10:96–115, 1927.

[13] G. L. Miller, S.-H. Teng, W. Thurston, and S. A. Vavasis. Geometric separators for finite-element meshes. *SIAM J. Sci. Comput.* 19(2):364–386, 1998.

[14] T. M. Przytycka and J. H. Przytycki. A simple construction of high representativity triangulations. *Discrete Math.* 173:209–228, 1997.

[15] R. Rado. A theorem on general measure. *J. London Math. Soc.* 21(4):291–300, 1946.

[16] D. Spielman and S.-H. Teng. Disk packings and planar separators. *Proc. 12th Ann. ACM Symp. Comput. Geom.*, 349–358, 1996.

[17] W. P. Thurston. *The Geometry and Topology of 3-Manifolds*. Princeton University lecture notes, 1980. ⟨http://www.msri.org/communications/books/gt3m⟩.