Strong duality

Announcement: Homework 4 due 11/3, 11:59 pm
Think about verifiers!

Take-home midterm will be 11/6 - 10.

RDK Weds office hrs. shifted 4-5 pm
this week + next.

\[
\begin{align*}
\text{max} & \quad 2x_1 + 3x_2 \\
\text{st.} & \quad x_1 + x_2 + w_1 = 8 \\
& \quad 2x_1 + x_3 + w_2 = 12 \\
& \quad x_1 + 2x_2 + w_3 = 14 \\
& \quad x, w \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad 12 + 3x_2 - w_2 \\
& \quad 8 - x_1 - x_2 \\
& \quad w_1 - 2 - \frac{1}{2}x_2 - \frac{1}{4}w_2 \\
& \quad x_1 = 6 - \frac{1}{2}x_2 - \frac{1}{4}w_2 \\
& \quad w_2 = 8 - \frac{3}{2}x_2 - \frac{1}{4}w_2 \\
& \quad x, w \geq 0
\end{align*}
\]

Suppose we start from \( x_2 = w_2 = 0 \) \( \iff \) new set of fixed variables.

In case \( x_2 \) from 0 to 4, which is when \( w_1 = 0 \) and \( x_1, w_3 \) are still \( \geq 0 \).

Finally rewrite everything as linear fun of \( w_1, w_2 \).

Iterate pivoting until one of the following things happens,

1. The objective function has a non-positive coeff. on every fixed variable.
   Then the current solution is certifiably optimal.

2. There's a fixed var. with positive coeff. in the objective function.
   For every non-fixed var, its partial derivative w.r.t. this fixed var is \( \geq 0 \).
   It means we found a ray contained in the feasible set on which the obj. function is unbounded.

Terminate with finite opt value.
Terminate and report opt is unbounded.
Example:  \[
\begin{align*}
\text{max} & \quad 18 - x_2 + w_3 \\
\text{subject to} & \quad x_1 = 5 + x_2 + \frac{1}{2} w_3 \\
& \quad w_1 = 3 - 4x_2 + 2w_3 \\
& \quad w_2 = 1 - \frac{1}{2}x_2 + \frac{1}{3}w_3
\end{align*}
\]

Degenerate points.

Example:  \[
\begin{align*}
\text{max} & \quad 18 - x_2 + w_3 \\
\text{subject to} & \quad x_1 = x_2 - w_3 \\
& \quad w_1 = 3 - 4x_2 + 2w_3 \\
& \quad w_2 = 1 - \frac{1}{2}x_2 + \frac{1}{3}w_3
\end{align*}
\]

Termination: means: objective function is now written as

\[
\text{obj} = v - z^T x - y^T w, \quad z, y \geq 0
\]

and we found a feasible point where

\[
z^T x = y^T w = 0.
\]

The equation \[c^T x = \text{obj} = v - z^T x - y^T w\]
means that \[c^T x = v - z^T x - y^T w\]
holds for all \(x, w\) satisfying \(A x + w = b\).
In other words
\[ c^T x = v - z^T x - y^T (b - Ax) \]
is valid \( x \in \mathbb{R}^n \).

\[ \therefore \quad 0 = v - y^T b \]
\[ c^T = -z^T + y^T A \]

We've got \( y, z \geq 0 \) s.t.
\[ v = b^T y \]
\[ A^T y = c + z \quad \Rightarrow \quad A^T y \notin c \]

Dual was
\[
\begin{align*}
\text{min} & \quad b^T y \\
\text{s.t.} & \quad A^T y \geq c \\
& \quad y \geq 0
\end{align*}
\]