The MAX-CUT problem is: given a graph $G$ (undirected) and integer $k > 0$, can the vertex set $V(G)$ be partitioned into $A, B$ such that at least $k$ edges are "cut" by the partition (meaning one endpoint in $A$, the other in $B$).

How to prove MAX-CUT is NP-hard?

A promising gadget for a 3SAT reduction, but it becomes tricky to represent clauses.

For reducing 3-COLORABILITY ≤P MAX-CUT...
Proposed Reduction: Given graph $H$ which is an instance of 3-Colorability, construct $G$ whose vertex set is:

- $\frac{1}{2} \times 7$ (root vertex)
- $\{u_0, u_1, u_2\}$ (3 gadget nodes corresponding) to each $u \in V(H)$

and edge set is

- Edges from $X$ to every other vertex
- Edges $(u_i, u_j)$ for all $u \in V(H)$ and $i \neq j$ in $\{0, 1, 2, 3\}$.
- Edges $(u_i, v_i)$ for all $(u, v) \in E(H)$ and $i \in \{0, 1, 2\}$.

Finally, we ask if $I$ a cut of size $4n + 2m$ edges.  ($n = \#$ whites in $H$, $m = \#$ edges in $H$).

Conclusion:

- Cuts that cut $> 4n$ black edges don't exist.
- Cuts that cut exactly $4n$ black edges and at least $2m$ purple edges do correspond to proper 3-colorings of $H$.
- Cuts that cut $< 4n$ black, but $\geq 4n + 2m$ edges in total may exist.
violating correctness of reduction.

If black edges have weight $w > 3m$ (total # purple edges)
then cuts of total weight $4nw + 2m$ must cut $4n$ black edges and,
in addition, at least $2m$ purples.

**Conclusion 2.** There is a slightly more general problem, \textsc{Weighted MAX CUT},
where the input is:

- graph $G$
- edge weights $w(u,v) \in \mathbb{N}$
- target cut weight $K$

and the question is: does $\exists$ a
vertex partition $A,B$ st.
\[ \sum_{u \in A, v \in B} w(u,v) > K ? \]

Then the reduction above, with
black edge weights set to $w = 3m+1$
and purple edge weights set to $1$
and $K = 4nw + 2m$, is a valid reduction
\textsc{3-Colorability} $\leq$ \textsc{Weighted MAX-CUT}
If we allow multigraphs (i.e. where two vertices can potentially be connected by multiple edges) then we can represent an edge of weight $w$ by $w$ parallel edges, and the reduction above becomes a reduction $3$-COLORABILITY $\leq_p$ MULTIGRAPH MAX-CUT.

If we want our reduction to output a simple graph, we require one more gadget. As before, let

$H = \text{instance of } 3$-COLORABILITY

$n = \# \text{ vertices of } H$

$m = \# \text{ edges of } H$

$w = 3m+1$

and now set

$M = 6nw + 3m + 1$

Replace each vertex (other than $x$) in the original reduction with $2M$ vertices forming a complete
bipartite graph with $M$ vertices on each side.

Thus, our reduction takes $H$ and outputs a graph $G$ with

- root vertex $X$
- vertices $u_{i;j}$ for all $u \in V(H)$, $i \in \{0, 1, 2\}$, $j \in \{1, 2, 3, ..., 2M\}$
- blue edges $(u_{i;j}, u_{i;k})$
  whenever $u \in V(H)$, $i \in \{0, 1, 2\}$, $j-k$ odd
- black edges $(u_{i;j}, u_{l;j})$ for $i \in \{0, 1, 2\}$ and $1 \leq j \leq w$
- black edges $(u_{i;j}, u_{l;j})$ for distinct $i, l \in \{0, 1, 2\}$ and $1 \leq j \leq w$
- purple edges $(u_{i;4}, v_{i;4})$
  whenever $(u_i, v_i) \in E(H)$ and $i \in \{0, 1, 2\}$.

This graph $G$ has $3nM^2$ blue edges,
6nw black edges, 3m purple edges.

MAX-CUT: Is there a partition that cuts at least
$$k = 3nM^2 + 4nw + 2m$$ edges of $G$?
Def. A partition of the set
$$V_{u,i} = \{ u_{ij} \mid 1 \leq j \leq 2m^2 \}$$
into sets $$A_{ui}, B_{ui}$$ is "pure"

if either
$$A_{ui} = \{ u_{ij} \mid j \text{ odd} \}, \quad B_{ui} = \{ u_{ij} \mid j \text{ even} \}$$
or
$$A_{ui} = \{ u_{ij} \mid j \text{ even} \}, \quad B_{ui} = \{ u_{ij} \mid j \text{ odd} \}$$

A partition of $$V(G) = \{ x_j \mid u_{ij} \in V_{u,i} \}$$
into sets $$A, B$$ is pure if

the partition $$i = A \cup V_{ui} \cup B \cup V_{ui} \cup V$$ is pure for all $$u, i$$.

Lemma. A pure partition cuts $$3nM^2$$ blue edges. Any other partition cuts at most $$3nM^2 - M$$ blue edges.

Proof. Every blue edge has endpoints $$u_{ij}$$ and $$u_{ik}$$ for some $$u \in V(H)$$,
$i \leq 2i, 2j$, and $j \in [2M]$ with $j-k$ odd. By definition a pure partition cuts every such edge.

If a partition $A, B$ is not pure, then there is $j,k$ such that $(A_{ij}, B_{ij})$ is not a pure partition of $V_{ij}$. If one of $A_{ij}, B_{ij}$ is empty, then none of the blue edges in $G[V_{ij}]$ is cut, so at least $M^2$ blue edges are uncut. If $A_{ij}, B_{ij}$ are both non-empty but $(A_{ij}, B_{ij})$ is not pure, then at least one of the sets $\{u_{ij} \mid j \text{odd} \}$ or $\{u_{ij} \mid j \text{even} \}$ intersects both $A_{ij}$ and $B_{ij}$.

Assume WLOG that it is $\{u_{ij} \mid j \text{odd} \}$. There are $M$ vertices in $u_{ij} \mid j \text{even} \}$ and each of them has at least $1$ blue edge to another vertex on its side of the partition. Hence at least $M$ blue edges...
Back to analyzing the reduction.

If $H$ is 3-colorable and $c: V(H) \to \{0,1,2\}$ is a valid coloring

let

$$A = \{x \notin V(H) \mid u \in V(H), i=(lu), \ j \ \text{even} \}
\cup \{u_{ij} \mid u \in V(H), i=(lu), \ j \ \text{odd} \}
\cup \{u_{ij} \mid u \in V(H), i \neq (lu), \ j \ \text{odd} \}$$

$$B = V(G) \setminus A$$

Then the partition $(A, B)$ cuts:

- all blue edges
- 4w black edges in each induced subgraph $G[ \{x \notin V(H) \cup V_0 \cup V_2 \cup V_2 \}, \ u \in V(H)]$
- 2m purple edges.

hence $K$ edges in total.
Conversely, if \( \exists \) a partition \((A, B)\) that cuts \(K\) edges in total:

1. The partition must be pure. Otherwise there are at least \(M\) uncult blue edges. By the choice of \(M > 6n_n + 3m\) the combined number of cut black and purple edges can't possibly compensate the loss of \(M\) blue edges from the cut.

2. For every \( u \in V(H) \), exactly one of \( u \) is \((0, 1, 2)\) satisfies \( \{u, j, j\} \in e\). Otherwise at most \(3w\) of the black edges in induced subgraph

\[
G[\{x \in V_{u0} \cup V_{u1} \cup V_{u2}\}]
\]

are cut. By choice of \(w > 3m\), no number of cut purple edges can compensate the loss of \(w\).
black edges from the cut.

3) Color each vertex $u \in V(H)$ with the unique color $c(u)=i$ such that $\{u, j\} \in \mathcal{E}$ with $i$ even. $\mathcal{E} \subseteq A$.

This must constitute a proper coloring of $H$, as otherwise fewer than $2m$ purple edges would be cut.