To show a problem $A$ is NP-complete you need two poly-time algorithms.

1) Verifier $V$: takes an input of $A(x)$ and a proposed solution $(y)$, and verifies $y$ is a valid solution to $x$. $A(x) = 1 \iff \exists y \ V(x,y) = 1$.

E.g. for 3SAT $V(x,y)$ takes logical formula $x$ and truth assignment $y$, outputs 1 if $y$ satisfies $x$.

2) Reduction $R$: takes an input $x_0$ of some other problem known to be hard, e.g. 3SAT, and transforms $x_0$ into an input of $A$.

$3SAT(x_0) = 1 \iff A(x_1) = 1$.

**Example:** INDEPENDENT SET.

Input: undirected graph $G$, positive integer $k$.
Output: 1 if and only if $I$ a subset of $k$ vertices of $G$, s.t. no edge has both of its endpoints in the subset.

E.g. \[
\begin{array}{c}
\text{2} \\
\end{array}
\] $\Rightarrow 1$ (yes there is a 2-element ind. set).
Why NP-Complete?

1. \( V(x,y) \) takes \( x = (G,k) \) and \( y = \) binary string encoding subset of \( V(G) \), say \( S \).

   - It checks \( |S| = k \). \( \mathcal{O}(1) \)
   - For each edge \( (u,v) \) it checks \( u \notin S \) or \( v \notin S \). \( \mathcal{O}(m) \)

2. We will reduce 3SAT to IND SET.

A structure that looks like a Boolean variable, translated to IND SET:

- 1-element independent subsets of this graph are in 1:1 correspondence with truth assignments of \( x_i \).

Selecting truth assignments for \( n \) variables:

\[
\begin{align*}
    &x_1 = 1 \quad x_1 = 0 \\
    &x_2 = 1 \quad x_2 = 0 \\
    &x_3 = 1 \quad x_3 = 0 \\
    &x_4 = 1 \quad x_4 = 0
\end{align*}
\]

\( \leftarrow \) \( n \) element independent subset of this graph are in 1:1 correspondence with truth assignments of \( x_1, \ldots, x_n \).

Adjoin a gadget like this for every clause. Set \( k = (\# \text{ vars}) + \# \text{ clauses} \).
IND SET restricted to graphs of max degree 3. 
(d3-IND-SET).

Gadget for a variable \(x_i\) that belongs to \(S\) clauses

Make each variable \(x_i\) into a gadget
with \(2n_i\) vertices & edges forming an
even cycle, where \(n_i\) denotes # clauses
containing \(x_i\) or \(\overline{x_i}\).

Set \(b = (\sum_i n_i) + (\# \text{ clauses})\)

GRAPH 3-COLORABILITY: Given undirected \(G\),
can we color its vertices with 3 colors
such that the endpoints of every edge
are differently colored.