**PUSH-RELABEL ALGORITHM**

initialize \( h(s) = n, \quad h(r) = 0 \) \( \forall v \notin S. \)

initialize \( f(u) = \begin{cases} c(u,v) & \text{if } u = s \\ -c(u,v) & \text{if } v = s \\ 0 & \text{otherwise} \end{cases} \)

while \( S \) \# of vertices with \( x(v) > 0 \):

let \( u \) be a vertex of positive excess with maximum height among all such vertices.

if \( \exists \) edge \( (u,v) \) with \( c(u,v) > f(u,v) \) and \( h(u) > h(v) \):

// PUSH \((u,v)\)

let \( \delta = \min \{ x(u), c(u,v) - f(u,v) \} \)

\[ f(u,v) \leftarrow f(u,v) + \delta \]

\[ f(v,u) \leftarrow f(v,u) - \delta \]

Saturating push

\[ \delta = c(u,v) - f(u,v) \]

non-saturating push

else:

// RELABEL \((u)\)

\[ h(u) \leftarrow h(u) + 1 \]

endwhile

output \( f \).

**Lemma.** If \( f \) is a preflow and \( v_0 \in S \) is a vertex with \( x(v_0) > 0 \) then \( G_f \) contains a path from \( v_0 \) to \( s \) made up of edges with \( c(u,v) - f(u,v) > 0 \).

**Proof.** Let \( E^+_f = \{ (u,v) \mid c(u,v) - f(u,v) > 0 \} \)

and \( A = \{ w \mid \text{there is a path from } w \text{ to } s \text{ with edges in } E^+_f \} \)

\[ B = V \setminus A. \]

Lemma asserts \( v_0 \in A \)
by construction \( A(x) \in G^+ \text{ with } u \in B, v \in A. \)

\[
\sum_{v \in B} x(u) = \sum_{v \in B} \sum_{u \in V} f(u, v) = \sum_{v \in A} \sum_{u \in B} f(u, v) + \sum_{u \in B} \sum_{v \in V} f(u, v)
\]

By skew symmetry

\[
\sum_{v \in B} f(v, u) \leftarrow \text{ Edges from } B \text{ to } A \text{ are saturated.}
\]

\[
\sum_{v \in B} \sum_{u \in A} c(v, u) \leq 0
\]

\( s \in A \), so \( x(u) \geq 0 \) for all \( v \in B \).

But \( \sum_{v \in B} x(u) \leq 0 \) so must be that \( x(u) = 0 \ \forall v \in B \).

\( \therefore v_0 \in A \) because \( x(v_0) > 0 \).

a.e.

Cor. At all times \( \forall v, h(v) < 2n. \)

Proof. The most recent time we relabeled \( v \),

\( x(u) \) was \( > 0 \). By lemma,

\( f \) had a path from \( v \) to \( s \)

with pos. resid. cap. on every edge.

\[
\begin{array}{c}
\circ \rightarrow (u) \rightarrow \circ \rightarrow \circ \rightarrow \circ \rightarrow (s) \\
\leq 2n-2 \leq 2n-2 \ldots \leq h_2 \leq n+1 \quad h(s) = n \\
\end{array}
\]

or else we wouldn't have relabeled.

\[
= \text{ relabels} \leq (2n-1)n.
\]
When $\text{PUSH}(u,v)$ saturates $(uv)$ it means residual cap. of $(uv)$ becomes equal to zero.
Also, $h(u) = h(v) + 1$.
We will not $\text{PUSH}(u,v)$ again until $(uv)$ has positive residual cap, which happens after $\text{PUSH}(v,u)$.

Between any two saturating pushes of $(u,v)$
$h(u)$ increases by 2 at most in saturating pushes per (oriented) edge.
$\leq 2mn$ saturating pushes total.

Non-saturating pushes

Let $H = \max$ over $v:\ x(v) > 0$ of $\{h(v)\}$.

Divide execution into phases when $H$ is constant.

$H$ can only increase during relabels. ($\leq Qn - 1)n$ times)

# phases $\leq 2(2n - i)n$.

Vertices $v$'s will be the source of a non-saturating push at most once per phase.

$\Rightarrow \leq 2(2n - i)n(n - i)$ non-saturating pushes.