Announcements:

1. Homework 3 to be released Fri; due a week from Fri, shorter than usual.

2. Email me and Shawn (rok2, so396) if you must switch groups.

3. In class midterm: I’m considering replacing it with a take-home, the week of Nov 6-10. (Flexible start/end.) Please email me if you’re not happy about this.

Recap. Flow satisfies $f(u, v) + f(v, u) = 0$

$$\sum_{v \in V} f(u, v) = 0 \quad \forall u \not\in \{s, t\}.$$ 

"Feasible" $f(u, v) \leq c(u, v) \quad \forall (u, v)$

Elementary Flow $f^e$: $s \rightarrow t$ with $\sum_{e \in E} f^e = f(s, t)$

Residual graph $G_f$, $f$ feasible $fBu$ in $G_f$.

Capacities $c_f(u, v) = c(u, v) - f(u, v)$

Augmenting path: $p$ from $s$ to $t$ whose edges have strictly positive residual capacity.

Def. An $s$-$t$ cut in flow network $G_r(V, s, t, c)$ is a partition of $V$ into $S, T$ with $s \in S$, $t \in T$.

For vertex sets $Q, R$ let

$$f(Q, R) = \sum_{u \in Q} \sum_{v \in R} f(u, v) \quad \text{net flow } Q \to R$$

$$c(Q, R) = \sum_{u \in Q} \sum_{v \in R} c(u, v) \quad \text{aggregate capacity } Q \to R$$
Observe:
(a) \( R_1, R_2 \) disjoint \( \Rightarrow f(Q, R_1 \cup R_2) = f(Q, R_1) + f(Q, R_2) \)
(b) \( f(Q, Q) = 0 \quad \forall Q \in V \)

... by skew-symmetry.

**Lemma.** If \( f \) is any flow and \( ST \) is any st cut,
\[ f(S, T) = \text{val}(f) \]

If \( f \) feasible,
\[ \text{val}(f) \leq c(S, T) \]

and equality holds if and only if \( f(uv) = c(uv) \)
for all \( u \in S \), \( v \in T \). ("\( S, T \) is saturated by \( f \).")

**Proof:** By properties (a), (b) above,
\[ f(S, T) = f(S, T) + f(S, S) = f(S, TV S) = f(S, V) \]
\[ = \sum_{u \in S} \sum_{v \in V} f(uv) \quad \text{inner sum equals zero except when } u = s. \]
\[ = \sum_{u \in V} f(S, V) = \text{val}(f). \]

Inequality \( \text{val}(f) \leq c(S, T) \) follows from for \( u \in S, v \in T \)
\[ \forall uv \quad f(uv) \leq c(uv) \]
\[ \text{this is strict} \]
\[ f(S, T) = \sum_{u \in S} \sum_{v \in T} f(uv) \leq \sum_{u \in S} \sum_{v \in T} c(uv) = c(S, T) \]

### Theorem (Max-flow Min-cut)
For a flow network \( G \) and a feasible flow \( f \), TFAE:

(i) \( f \) is a maximum flow
(ii) there is no augmenting path in \( G \)
(iii) there exists a st cut with \( c(S, T) = \text{val}(f) \)
(iv) there exists a minimum s-t cut with \( c(S, T) = \text{val}(f) \)

**Proof.** First (iv) \( \Rightarrow \) (iii) obvious. To prove (iii) \( \Rightarrow \) (iv)
assume \( f, S, T \) satisfy (iii) and assume
$S^*, T^*$ is any $s^t$ cut of minimum capacity.

\[ c(S^*, T^*) \leq c(S, T) \quad \text{(def of } S^*, T^*) \]

\[ c(S, T) = \text{val}(F) \leq c(S^*, T^*) \]

Hence $c(S^*, T^*) = c(S, T)$ are equal, so $S'T$ is a minimum $s^t$ cut satisfying $\text{val}(F) - c(S, T)$ as required by (iv).

For (i) $\Rightarrow$ (ii) we have $(-ii) \Rightarrow (-i)$.

If $G_f$ has any path $P$, let

\[ s(P) = \min \{ \text{clm}(u) - \text{flx}(u) \mid (u) \text{ an edge of } P \} \]

Then $f + s(P): f^P$ is also a feasible flow, its value is $\text{val}(F) + s(P) > \text{val}(F)$, so $f$ is not a max flow.

For (ii) $\Rightarrow$ (iii): define an augmenting walk to be a sequence $S = \{u_0, u_1, u_2, \ldots, u_k\}$ of vertices, such that residual $\text{cap} > 0$ for all $(u_i, u_{i+1}), 0 \leq i < k$, $c(u_i, u_{i+1}) = \text{flx}(u_i, u_{i+1})$.

Let $S = \{u\}$ be an augmenting walk ending at $u^2$.

$T = V \setminus S$.

Note $s \in S$ because $(S)$ is augmenting walk.

$u \in T$ because $S$ is augmenting path in $G_f$.

Every $(u, v)$ with $u \in S, v \in T$ has zero residual capacity. This is because any augmenting walk $S = \{u_0, u_1, \ldots, u_k = v\}$ but $u_0, u_1, \ldots, u_k, u_{k+1} = v$ is not an augmenting walk. $\Rightarrow (u, v)$ has $0$ residual capacity. $\therefore = 0$.

We have an $s^t$ cut which is saturated by $f$, so $c(S, T) = \text{val}(F)$ by Lemma above.
Lastly, for $(iii) \Rightarrow (i)$:

If $F$ is feasible flow, $S,T$ is st cut and $\text{val}(F) = c(S,T)$, then for any feasible flow $f^*$,

$$\text{val}(f^*) \leq c(S,T) = \text{val}(F)$$

by Lemma

$\therefore \text{val}(F)$ is the maximum value of a feasible flow in $G$. 
