Announcements:

1. Lecture notes to be posted in a few days.
2. NO CLASS Mon 9/25.

What does a flow look like?

Def. A flow is a function $f : V \times V \to \mathbb{R}$ such that:

- skew symmetry: $f(u,v) + f(v,u) = 0 \quad \forall u,v$
- flow conservation: $\sum_{v \in V} f(u,v) = 0 \quad \forall u \neq s,t$

The value of a flow, $\text{val}(f)$, is

$\text{val}(f) = \sum_{v \in V} f(s,v)$.

A flow network $(V, s, t, c)$ is a vertex set $V$,
source $s$, sink $t$, and capacity function

$c : V \times V \to [0, \infty]$.

Flow $f$ is feasible if $f(u,v) \leq c(u,v) \quad \forall u,v \in V$. 
The maximum s-t flow problem is:
given \((V, s, t, c)\) find a feasible flow \(f\) maximizing \(\text{val}(f)\). (If one exists, i.e. \(\sup \{\text{val}(f)\} < \infty\)).

A flow is a "weighted sum of paths and cycles".

**Def.** Suppose \(P = V_0, V_1, \ldots, V_k\) is a sequence in \(V\) and either

(i) \(P\) is a simple path with endpoints \(s, t\):
\[\{V_0, V_k\} = \{s, t\}\] and \(V_0, \ldots, V_k\) distinct

(ii) \(P\) is a simple cycle
\[V_0 = V_k\] and \(V_0, \ldots, V_k\) distinct.

Then the elementary flow \(f^P\) assoc to \(P\) is

\[f^P(u, v) = \begin{cases} 1 & \text{if } \exists i \ u = V_i, \ v = V_{i+1} \\ -1 & \text{if } \exists i \ v = V_i, \ u = V_{i+1} \\ 0 & \text{otherwise}. \end{cases}\]

**Lemma.** Every flow is a non-negative weighted sum of elementary flows, \(f = \sum_P \omega_P f^P\).

And \(\text{val}(f) = \left(\sum_{P: s \to t} \omega_P\right) - \left(\sum_{P: t \to s} \omega_P\right)\) in any such decomposition.
Def.: If \( f \) is a feasible flow in network \( (V, s, t, c) \) the residual network of \( f \) is \( G_f = (V, s, t, c-f) \).

Lem.: If \( f \) is a feasible flow in \( G = (V, s, t, c) \) there is a bijection

\[
\left\{ \text{feasible flows} \right\} \leftrightarrow \left\{ \text{feasible flows} \right\}
\]

\[
f' \text{ in } G \leftrightarrow h \text{ in } G_f
\]

\[
f' \quad \leftrightarrow \quad f' - f
\]

\[
h + f \quad \leftrightarrow \quad h
\]

\( \forall u, v \ h(u,v) \leq c(u,v) \)

This bijection is value-preserving up to an additive shift of \( \pm \text{val}(f) \).

Cor.: \( f \) is a max flow in \( G \) if and only if \( 0 \) is a max flow in \( G_f \).

If \( G_f \) contains a path \( P \) from \( s \) to \( t \) made up of edges with capacity \( > 0 \), let \( \delta = \min \{ c(u,v) - f(u,v) \mid (uv) \text{ an edge of } P \} \) and observe \( \delta \cdot f^P \) is feasible in \( G_f \),

\[
\text{val}(\delta \cdot f^P) = \delta > 0 \quad \Rightarrow \quad f \text{ is not a max flow in } G.
\]
Connecting max-flow with bipartite matching.

Instance of max bipartite matching

Flow network